Non-Commutative U(1) Gauge Theory on R**4 with Oscillator Term and BRST Symmetry

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action

We suggest the following gauge fixed action in the classical limit [1]:

$$\Gamma^{(0)} = \Gamma_{\text{inv}} + \Gamma_{\text{m}} + \Gamma_{\text{gf}}\,,$$
$$\Gamma_{\text{inv}} = \frac{1}{4}\int d^4x\, F_{\mu\nu}\star F_{\mu\nu}\,,$$
$$\Gamma_{\text{m}} = \frac{\Omega^2}{4}\int d^4x \left(\frac{1}{2}\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\star\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} + \{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\}\star\{\tilde{x}_\mu \stackrel{\star}{,} c\}\right) = \frac{\Omega^2}{8}\int d^4x\,(\tilde{x}_\mu \star \mathcal{C}_\mu)\,,$$
$$\Gamma_{\text{gf}} = \int d^4x \left[B\star\partial_\mu A_\mu - \frac{1}{2}B\star B - \bar{c}\star\partial_\mu sA_\mu - \frac{\Omega^2}{8}\widetilde{c}_\mu \star s\,\mathcal{C}_\mu\right]$$

with

$$\mathcal{C}_\mu = \Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\}\stackrel{\star}{,} c] + [\bar{c}\stackrel{\star}{,}\{\tilde{x}_\mu \stackrel{\star}{,} c\}]\Big)$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} A_\nu]\,,$$
$$\tilde{x}_\mu = \left(\theta^{-1}\right)_{\mu\nu} x_\nu,$$
$$\mathrm{i}\theta_{\mu\nu} = [x_\mu \stackrel{\star}{,} x_\nu]\,.$$

BRST symmetry

The action is invariant under the BRST transformations given by

$$sA_\mu = D_\mu c = \partial_\mu c - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} c]\,, \qquad s\bar{c} = B,$$
$$sc = \mathrm{i}gc\star c, \qquad sB = 0,$$
$$s\widetilde{c}_\mu = \tilde{x}_\mu, \qquad s^2\varphi = 0\ \forall\ \varphi\in\{A_\mu, B, c, \bar{c}, \widetilde{c}_\mu\}\,.$$

Furthermore, the fermionic multiplier field $\widetilde{c}_\mu$ imposes an additional constraint, namely on-shell BRST invariance of $\mathcal{C}_\mu$ and hence of the mass terms Γ_{m}.

Feynman rules

The bilinear parts of the action lead to the following improved propagators:

$$G^{A}_{\mu\nu}(x-y) = \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1}\delta_{\mu\nu}\delta^4(x-y),$$
$$G^{\bar{c}c}(x-y) = \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1}\delta^4(x-y),$$

$$G^{BA}_{\mu}(x-y) = \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1}\partial_\mu\delta^4(x-y),$$
$$G^{B}(x-y) = \left[\partial_\mu\left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1}\partial_\mu - 1\right]\delta^4(x-y).$$

Both the gauge field and the ghost propagators are essentially the Mehler kernel, which in momentum space reads

$$K_M(p,q) = \frac{\omega^3}{8\pi^2}\int_0^\infty d\alpha\frac{1}{\sinh^2(\alpha)}\exp\left(-\frac{\omega}{4}u^2\coth\frac{\alpha}{2} - \frac{\omega}{4}v^2\tanh\frac{\alpha}{2}\right),$$
$$u = (p-q)\,, \qquad v = (p+q)\,,$$

so we may expect improved IR behaviour of the Feynman graphs. Since there are no vertices involving the B field and since the additional multiplier $\widetilde{c}_\mu$ has no propagator, neither field will play a role in loop corrections.
The important vertices of the model in momentum space are given by

$$\widetilde{V}^{3A}_{\rho\sigma\tau}(k_1,k_2,k_3) = 2\mathrm{i}g(2\pi)^4\delta^4(k_1+k_2+k_3)\,[(k_3-k_2)_\rho\delta_{\sigma\tau} + (k_1-k_3)_\sigma\delta_{\rho\tau} + (k_2-k_1)_\tau\delta_{\rho\sigma}]\sin\left(\frac{k_1\times k_2}{2}\right),$$

$$\begin{aligned}\widetilde{V}^{4A}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4) = {}& -4g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\\ &\times\Bigg[(\delta_{\rho\tau}\delta_{\sigma\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)\\ &+(\delta_{\rho\sigma}\delta_{\tau\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)\\ &+(\delta_{\rho\sigma}\delta_{\tau\epsilon} - \delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\frac{k_2\times k_3}{2}\right)\sin\left(\frac{k_1\times k_4}{2}\right)\Bigg],\end{aligned}$$

$$\widetilde{V}^{c}_{\mu}(q_1,k_2,q_3) = -2\mathrm{i}g(2\pi)^4\delta^4(q_1+k_2+q_3)q_{3\mu}\sin\left(\frac{q_1\times q_3}{2}\right),$$

equations of motion

$$\frac{\delta\Gamma^{(0)}}{\delta B} = \partial_\mu A_\mu - B + \frac{\Omega^2}{8}\Big([\{\tilde{x}_\mu \stackrel{\star}{,} c\}\stackrel{\star}{,}\widetilde{c}_\mu] - \{\tilde{x}_\mu \stackrel{\star}{,} [\widetilde{c}_\mu \stackrel{\star}{,} c]\}\Big) = 0$$

$$\begin{aligned}\frac{\delta\Gamma^{(0)}}{\delta A_\nu} = {}& \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)A_\nu + \mathrm{i}g\left([A_\mu \stackrel{\star}{,} F_{\mu\nu}] + \partial_\mu[A_\mu \stackrel{\star}{,} A_\nu] + \{\partial_\nu\bar{c}\stackrel{\star}{,} c\}\right)\\ &+ \partial_\nu(\partial A) - \partial_\nu B + \frac{\Omega^2}{8}\Big(\{[D_\nu c\stackrel{\star}{,}\widetilde{c}_\mu]\stackrel{\star}{,}\tilde{x}_\mu\} + [\{D_\nu c\stackrel{\star}{,}\tilde{x}_\mu\}\stackrel{\star}{,}\widetilde{c}_\mu]\Big)\\ &- \mathrm{i}g\frac{\Omega^2}{8}\Big(\{c\stackrel{\star}{,}\{\tilde{x}_\mu \stackrel{\star}{,}\{A_\nu \stackrel{\star}{,}\widetilde{c}_\mu\}\}\} + \{c\stackrel{\star}{,}\{\widetilde{c}_\mu \stackrel{\star}{,}\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\}\}\Big) = 0\end{aligned}$$

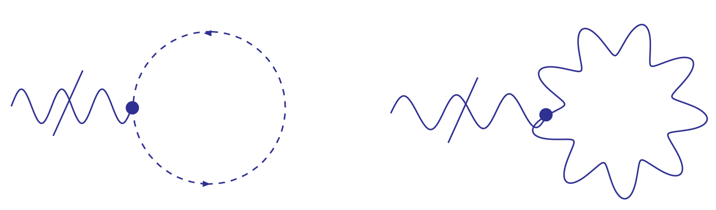
$$\begin{aligned}\frac{\delta\Gamma^{(0)}}{\delta\bar{c}} = {}& \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)c + \mathrm{i}g\partial_\mu[A_\mu \stackrel{\star}{,} c]\\ &- \mathrm{i}g\frac{\Omega^2}{8}\Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} c\star c\}\stackrel{\star}{,}\widetilde{c}_\mu\} + \{\tilde{x}_\mu \stackrel{\star}{,}\{\widetilde{c}_\mu \stackrel{\star}{,} c\star c\}\}\Big) = 0\end{aligned}$$

$$\begin{aligned}\frac{\delta\Gamma^{(0)}}{\delta c} = {}& \left(\Delta_4 - \Omega^2\tilde{x}^2\right)\bar{c} + \frac{\Omega^2}{8}\Big(\{\widetilde{c}_\mu \stackrel{\star}{,}\{\tilde{x}_\mu \stackrel{\star}{,} B\}\} + \{\tilde{x}_\mu \stackrel{\star}{,}\{\widetilde{c}_\mu \stackrel{\star}{,} B\}\}\Big)\\ &- \mathrm{i}g\,[A_\mu \stackrel{\star}{,}\partial_\mu\bar{c}] - \frac{\Omega^2}{8}D_\nu\Big(\{\tilde{x}_\mu \stackrel{\star}{,}\{A_\nu \stackrel{\star}{,}\widetilde{c}_\mu\}\} + \{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\stackrel{\star}{,}\widetilde{c}_\mu\}\Big)\\ &+ \mathrm{i}g\frac{\Omega^2}{8}\Big([c\stackrel{\star}{,}[\widetilde{c}_\mu \stackrel{\star}{,}\{\tilde{x}_\mu \stackrel{\star}{,}\bar{c}\}]] - [c\stackrel{\star}{,}\{\tilde{x}_\mu \stackrel{\star}{,}[\bar{c}\stackrel{\star}{,}\widetilde{c}_\mu]\}]\Big) = 0\end{aligned}$$

$$\frac{\delta\Gamma^{(0)}}{\delta\widetilde{c}_\mu} = -\frac{\Omega^2}{8}s\,\mathcal{C}_\mu = 0$$

properties of the star product

$$A_1(x)\star A_2(x) = e^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu}A_1(x)A_2(y)\Big|_{x=y},$$
$$\int d^4x A_\mu(x)\star A_\nu(x)\star A_\rho(x) = \int d^4x A_\rho(x)\star A_\mu(x)\star A_\nu(x),$$
$$\int d^4x A_\mu(x)\star A_\nu(x) = \int d^4x A_\mu(x)A_\nu(x),$$
$$\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu(x)\} = 2\tilde{x}_\mu A_\nu(x).$$



gauge field tadpole with amputated external leg

- $\Omega \neq 0$

$$\begin{aligned}T_\rho &= \int d^4k\int d^4k'\,\tilde{K}_M(k,k')V^{3A}_{\rho\sigma\tau}(p,-k',k)\delta_{\sigma\tau}\\ &= 6i\pi^4 g\,\tilde{p}_\mu\int d\alpha\frac{\sinh\alpha}{(\cosh\alpha-1)^3}\\ &\quad\times\exp\left(\frac{\omega}{4}p^2\frac{\cosh\alpha-1}{\sinh\alpha} - p^2\frac{\theta^2}{16\,\omega}\frac{\sinh\alpha}{\cosh\alpha-1}\right)\\ &= \text{finite}\end{aligned}$$

- $\Omega = 0$

$$\begin{aligned}T_\rho &= \sum_{\eta=\pm1} i\eta^2 3(2\pi)^4\pi^2 g\left(\delta^{(4)}(p)\tilde{p}_\rho\right)\int_0^\infty d\alpha\,\frac{1}{\alpha^3}\exp\left\{-\frac{\tilde{p}^2}{4\alpha}\right\}\\ &= 6i(2\pi)^4\pi^2 g\left(\delta^{(4)}(p)\tilde{p}_\rho\right)\frac{16}{\tilde{p}^4}\ \rightarrow\ \infty\end{aligned}$$

open questions

- The bilinear parts of the action are Langmann-Szabo invariant, but some vertices are not and the question remains whether this is sufficient to remove UV/IR mixing problems.
  Loop calculations are currently work in progress.
- What are the (classical) solutrions to the equations of motion?
- Is there a connection between this model and induced gauge theory [2,3].

references

[1] D. N. Blaschke, H. Grosse and M. Schweda, *Non-Commutative U(1) Gauge Theory on R**4 with Oscillator Term and BRST Symmetry*, *Europhys. Lett.* **79** (2007) 61002, [arXiv:0705.4205].

[2] H. Grosse and M. Wohlgenannt, *Induced gauge theory on a noncommutative space*, *Eur. Phys. J.* **C52** (2007) 435–450, [arXiv:hep-th/0703169].

[3] A. de Goursac, J.-C. Wallet and R. Wulkenhaar, *Noncommutative induced gauge theory*, *Eur. Phys. J.* **C51** (2007) 977–987, [arXiv:hep-th/0703075].