Quantum Field Theory on Non-Commutative Spaces

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Version: January 4, 2012

Foreword

These lecture notes provide an introduction to quantum field theories on non-commutative spaces. They are mainly based on the review articles [1–6]. In Chapter 1 we start with some motivations for studying such theories and introduce the so-called Landau problem and its relation to the quantum Hall effect (following [2, 7, 8]). We then recall some basic properties of C^*-algebras [1]. In Chapter 2 we deal with models with constant coordinate commutator. We mostly follow [2, 6] with amendments from [3, 4], except for Section 2.5 which follows [5, 9]. Chapter 3 finally deals with more general coordinate commutators and mostly follows the review article [6] with amendments from [10, 11] (Section 3.3), [12] (Section 3.4) and [13] (Section 3.6).

The interested reader is encouraged to consult the references mentioned above as well as the books [14, 15] for further details and references to the original literature.

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1 Introduction

1.1 Motivation

Many arguments can be given to motivate studying non-commutative geometry. The list below hence only represents a brief overview.

1. Historically, the idea of a "minimal length" was initially introduced in order to smear out point-like interactions as UV regularization in QFTs. The first according publication was due to H. Snyder, a former student of Oppenheimer, in 1946/47. However, it took a long time (i.e. until the early 1990s) until this idea was revived.

2. Coordinate non-commutativity appears in some cases, where one would not expect it at first: for example, when considering a point-like particle in a strong external magnetic field. This is known as the "Landau problem", which is in fact related to the quantum Hall effect, and will be discussed subsequently in Section 1.2.

3. Mathematically, General Relativity and QFT are incompatible as can be seen from the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \langle T_{\mu\nu} \rangle \ . \tag{1.1}$$

   On the left hand side we have the classical Einstein tensor describing space-time curvature induced by the energy momentum tensor $T_{\mu\nu}$. On the other hand, we know that matter is very well described by quantum field theory, and hence $T_{\mu\nu}$ should be an operator (hence the vev on the r.h.s. above). The situation is resolved by arguing that quantum effects can be neglected when considering long distances and replacing $T_{\mu\nu}$ with its classical counterpart. However, there definitely must be situations where both gravity and quantum effects will play a role, i.e. near black holes, so that a more general quantum theory of gravity is needed. Perhaps this argument represents the strongest hint towards quantized space-time.

Several theories of, or incorporating, quantum gravity exist today, although none of them could so far be verified by experiments. The ones that are most actively pursued in modern physics are String Theory, Quantum Loop Gravity, and Non-Commutative Quantum Field Theories. Here, we shall consider the latter approach.

At this point we should also mention some further pioneers of non-commutative geometry and quantum field theories thereon, namely Groenewold (1946), Moyal (1949), Madore (1992), Connes (1994), Filk (1996) and others. Additionally, people like Seiberg, Witten, Douglas, Schwarz and others studied non-commutative geometries in connection with string theory. For example, the effective low-energy action on D-branes with strong B-field background, can become non-commutative. In fact, the mechanism is not unlike the one in the Landau problem (cf. Section 1.2). In Section 2.4 we will study the so-called Seiberg-Witten map which dates back to 1998 and has its origin in the string theory ideas above. For further details and references on historic developments in this field, the reader is referred to the review articles [2–4, 6].

Having motivated space-time quantization, a natural question to ask is about the order of magnitude of an according uncertainty principle. A natural limit in the experimental length resolution can be given by the following argument which is due to Doplicher, Fredenhagen and Roberts (1994): Without specifying the required experimental setup, it is clear from quantum theory that a better length resolution always requires higher energy. In fact, the energy required for a resolution of the order of the Planck length turns out to have a Schwarzschild radius of the Planck length which is given by

$$\Delta x_\mu \simeq \lambda_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33}\mathrm{cm}\,. \tag{1.2}$$

Hence, distances beyond the Planck length can never be resolved experimentally, and it is natural to consider λ_p as a lower bound for a new uncertainty principle such as $\Delta x \Delta y \gtrsim \lambda_p^2$. An upper bound is given by the fact that we have not seen any effects of quantum space-time in experiments so far. Since this bound is constantly being lowered by new observations and some of these are passionately debated, we will not state an explicit order of magnitude at this point.

Exercise 1 *Consider an energy E whose de Broglie wavelength λ equals the Schwarzschild radius of E, and verify that λ must then be of the order of the Planck length* (1.2) *and E of the order of the Planck energy!*

1.2 The Landau-problem

We consider a charged particle in a constant magnetic field. The action is given by

$$S = -m\int ds + e\int A_\mu dx^\mu\,, \tag{1.3}$$

of which we consider the non-relativistic approximation $ds \approx (1-\frac{1}{2}\dot{\vec{x}}^2)dt$. Furthermore, we consider the form $A_i = -B/2\epsilon_{ij}x^j$ with $i,j \in [1,2]$ (and $A_0 = A_3 = 0$) for the vector potential A_μ in order to have the constant magnetic field B point in the x_3 direction. Hence, we obtain

$$S = \int dt\left(\frac{1}{2}m\dot{\vec{x}}^2 - e\vec{A}\dot{\vec{x}}\right)\,. \tag{1.4}$$

A Legendre transformation leads to the Hamiltonian

$$H(\vec{x},\vec{p}) = \vec{p}\dot{\vec{x}} - L(\vec{x},\dot{\vec{x}}) = \frac{1}{2m}\left(\vec{p}+e\vec{A}\right)^2\,, \tag{1.5}$$

where $\vec{p} = m\dot{\vec{x}} - e\vec{A}$ is the canonical momentum. Upon introducing the physically observable momentum $\vec{\pi} = m\dot{\vec{x}} = \vec{p}+e\vec{A}$, the Hamiltonian takes the particularly simple form $H = \frac{1}{2m}\vec{\pi}^2$.

When quantizing the system, i.e. when introducing canonical commutation relations $\left[\hat{x}^j,\hat{p}_i\right] = \mathrm{i}\delta_i^j$, we find that the physical momentum operators $\hat{\pi}_i$ have a non-vanishing quantum commutator

$$[\hat{\pi}_i,\hat{\pi}_j] = \mathrm{i}eB\epsilon_{ij}\,, \tag{1.6}$$

meaning that the momentum space in the presence of a background magnetic field B becomes non-commutative. Due to (1.6) we may define creation and annihilation operators as the linear combinations

$$\begin{aligned} a &= \frac{1}{\sqrt{2eB}}(\pi_1 + \mathrm{i}\pi_2)\,, \qquad a^+ = \frac{1}{\sqrt{2eB}}(\pi_1 - \mathrm{i}\pi_2)\,, \\ \left[a, a^+\right] &= 1\,. \end{aligned} \tag{1.7}$$

Hence we may once more rewrite the Hamiltonian as

$$H = \frac{eB}{m}\left(a^+ a + \frac{1}{2}\right)\,, \tag{1.8}$$

and now we can see that its spectrum is that of a harmonic oscillator, namely

$$E_n = \frac{eB}{m}\left(n + \frac{1}{2}\right)\,. \tag{1.9}$$

These energy eigenvalues are called the Landau levels.

Spacial non-commutativity arises in the limit $m \to 0$, i.e.

$$\left[\hat{x}^i, \hat{x}^j\right] = \frac{2\mathrm{i}}{eB}\epsilon^{ij}\,. \tag{1.10}$$

In this limit the theory becomes topological because the Hamiltonian vanishes and there are no propagating degrees of freedom. In fact, the limit $m \to 0$ above corresponds to the projection onto the lowest Landau level. Equivalently, one may consider a large magnetic field, i.e. $B \to \infty$.

Further insight can be gained by introducing "guiding center coordinates" $R^i = \frac{1}{2}x^i - \frac{1}{eB}\epsilon^{ij}p_j$ with the commutator given by

$$\left[R^i, R^j\right] = \frac{\mathrm{i}\epsilon^{ij}}{eB}\,, \qquad \left[\pi^i, R^j\right] = 0\,. \tag{1.11}$$

They can be combined into the two oscillators

$$b^+ = \frac{eB}{\sqrt{2}}(R_x + \mathrm{i}R_y)\,, \qquad b = \frac{eB}{\sqrt{2}}(R_x - \mathrm{i}R_y)\,, \tag{1.12}$$

and the lowest Landau level wave functions are obtained by acting with b^+ on the ground state of the according Hamiltonian (1.8).

The Quantum Hall Effect

The so-called integer quantum Hall effect can be observed experimentally in a setup as above with strong magnetic field B and very low temperature. It is then observed that the Hall resistance is quantized, i.e. as a function of n/B, where n denotes the electron number density, the off-diagonal components of the conductivity tensor σ_{ij} exhibit "steps": $\sigma_{12} = -\sigma_{21} = k\mathrm{i}e^2/h$ where k is an integer, namely the number of filled Landau levels. At each of these "plateaus" the diagonal elements $\sigma_{11} = \sigma_{22} = 0$ — see Figure 1.1.

Additionally, there is also the fractional quantum Hall effect which can be observed under the additional condition that the interactions between the electrons dominate over the effect

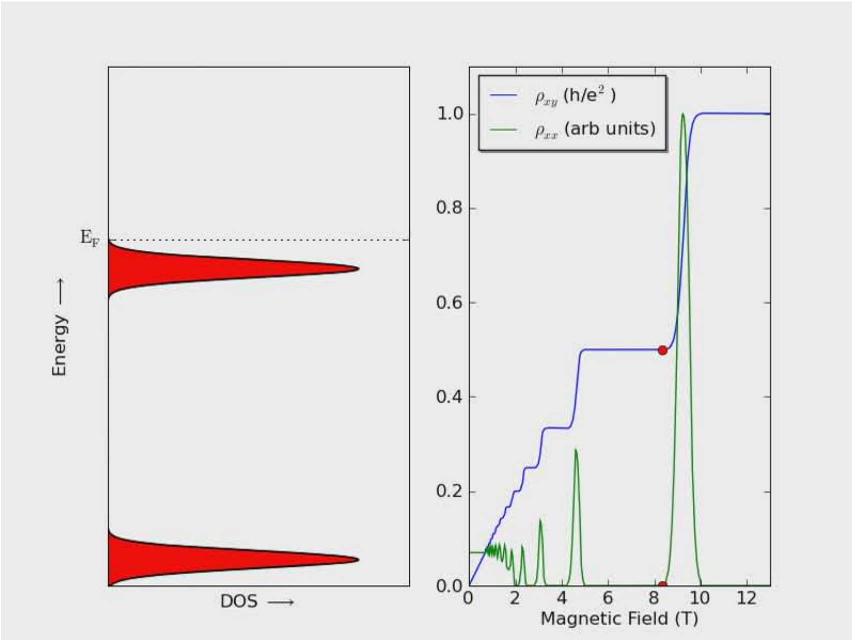


Figure 1.1: © Glenton Jelbert. The graph on the l.h.s. depicts Fermi energy vs. density of states while the one on the r.h.s. shows Hall resistance ρ_{xy} and resistance ρ_{xx} as functions of the magnetic field. (Note that the conductivity tensor σ_{ij} is given by the inverse of ρ_{ij}.)

of disorder. It is similar to the integer quantum Hall effect except that k takes specific rational values 1/3, 2/3, 2/5, ... (but no values with even denominators such as 1/2, 1/4), i.e. one observes "plateaus" at partially filled Landau levels. Especially the fractional quantum Hall effect is not fully understood to date, and one hopes that an explanation might be found using techniques from non-commutative geometry. In fact, as suggested by Susskind 2001, a description in terms of a non-commutative Chern-Simons model could be successful in this respect.

1.3 Non-commutative space-times and C^*-algebras

Let us specify what is meant by the notion of "non-commutative geometry". It is known that there is a correspondence between geometric spaces and commutative C^*-algebras, which is captured by the Gel'fand-Naimark theorem. Hence, a natural definition of a non-commutative geometry would be in terms of a non-commutative C^*-algebra. Furthermore, "points" are identified with pure states in non-commutative geometry since in such a setting the notion of a point no longer makes sense. We therefore start by reviewing the basic properties of a general C^*-algebra:

Let $\mathcal{A}$ be an algebra over the field of complex numbers $\mathbb{C}$ (i.e. think of a vector space over

$\mathbb{C}$) and a product $\mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ which is distributive over addition,

$$\begin{aligned}
&(a, b) \mapsto ab \in \mathcal{A}\,, \\
&a(b+c) = ab + ac\,, \\
&(a+b)c = ac + bc\,, \qquad \forall a, b, c \in \mathcal{A}\,.
\end{aligned} \tag{1.13}$$

The algebra $\mathcal{A}$ should also have a unit element denoted by $\mathbb{1}$. If $\mathcal{A}$ admits an anti-linear involution $* : \mathcal{A} \mapsto \mathcal{A}$ with the properties

$$\begin{aligned}
&a^{**} = a\,, && (ab)^* = b^* a^*\,, \\
&(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*\,, &&
\end{aligned} \tag{1.14}$$

for $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, and where bar denotes complex conjugation, one speaks of a $*$-algebra. We are furthermore interested in normed algebras, and hence consider a norm $\|\cdot\| : \mathcal{A} \mapsto \mathbb{R}$ having the properties

$$\begin{aligned}
&\|a\| \geq 0\,, && \|a\| = 0 \Leftrightarrow a = 0\,, && \|\alpha a\| = |\alpha| \|a\|\,, \\
&\|a+b\| \leq \|a\| + \|b\|\,, && \|ab\| \leq \|a\| \|b\|\,. &&
\end{aligned} \tag{1.15}$$

This defines the so-called norm topology, and corresponding neighbourhoods of $a \in \mathcal{A}$ are given by

$$U(a, \varepsilon) = \{ b \in \mathcal{A} \,|\, \|a - b\| < \varepsilon \}\,, \qquad \varepsilon > 0\,. \tag{1.16}$$

A Banach $*$-algebra is a normed $*$-algebra where $\|a^*\| = \|a\|$ and which is compact. Imposing the additional condition

$$\|a^* a\| = \|a\|^2\,, \qquad \forall a \in \mathcal{A}\,, \tag{1.17}$$

we finally arrive at a C^*-algebra.

A left ideal (resp. right ideal) $\mathcal{I}$ is a proper, norm closed subalgebra of $\mathcal{A}$ if $a \in \mathcal{A}$ and $b \in \mathcal{I}$ imply that $ab \in \mathcal{I}$ (resp. $ba \in \mathcal{I}$). In fact, any $*$-ideal (containing the $*$ of its elements) is automatically two-sided (i.e. left and right). Furthermore, if $\mathcal{A}$ is a C^*-algebra, then the quotient $\mathcal{A}/\mathcal{I}$ is also a C^*-algebra.

A $\mathbb{C}$-linear map $\pi : \mathcal{A} \mapsto \mathcal{B}$ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions

$$\pi(ab) = \pi(a)\pi(b)\,, \qquad \pi(a^*) = \pi(a)^*\,, \qquad \forall a, b \in \mathcal{A}\,, \tag{1.18}$$

is called a $*$-morphism, and if it is also bijective, one has a $*$-isomorphism.

A representation of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π is a $*$-morphism

$$\pi \,:\, \mathcal{A} \mapsto \mathcal{B}(\mathcal{H})\,, \tag{1.19}$$

and where $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of bounded operators of $\mathcal{H}$. It is called faithful if $\ker(\pi) = \{0\}$ so that π is an isomorphism, and irreducible if the only closed subspaces of $\mathcal{H}$ which are invariant under the action of $\pi(\mathcal{A})$ are the trivial subspaces $\{0\}$ and $\{\mathcal{H}\}$.

Finally, two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are equivalent if there exists a unitary operator $U \,:\, \mathcal{H}_1 \mapsto \mathcal{H}_2$ such that

$$\pi_1(a) = U^* \pi_2(a) U\,. \tag{1.20}$$

Now consider the special case of a commutative C^* algebra $\mathcal{A}$. In that case, clearly every irreducible representation is one-dimensional and hence $\pi : \mathcal{A} \mapsto \mathbb{C}$ with $\pi(\mathbb{1}) = 1$, $\forall \pi \in \hat{\mathcal{A}}$, where $\hat{\mathcal{A}}$ denotes the space of equivalence classes of irreducible representations of $\mathcal{A}$ called the structure space of $\mathcal{A}$. For example, suppose that the algebra $\mathcal{A}$ is generated by N-commuting self-adjoint[1] elements $x_1, \ldots, x_N$. Then the structure space $\hat{\mathcal{A}}$ can be identified with a compact subset of $\mathbb{R}^N$ by the map

$$\pi \in \hat{\mathcal{A}} \mapsto (\pi(x_1), \ldots, \pi(x_N)) \in \mathbb{R}^N . \tag{1.21}$$

For further details on C^*-algebras we refer the interested reader to the literature, especially [1] as well as [14, 15].

Having now very briefly introduced C^*-algebras, we proceed with an overview over various choices of non-commutative algebras generated by "coordinates" $\hat{x}^i$ which now are promoted to Hermitian operators (denoted by hats) on some Hilbert space $\mathcal{H}$. More precisely, we consider the algebra of formal power series divided by an ideal $\mathcal{I}$ of relations generated by the commutator of the coordinate functions $\left[\hat{x}^i, \hat{x}^j\right] \neq 0$, i.e.

$$\mathcal{A} = \frac{\mathbb{C}\langle \hat{x}^1, \ldots, \hat{x}^N \rangle}{\mathcal{I}} . \tag{1.22}$$

Commutators of coordinates

The commutator of the coordinates has the general form

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}(\hat{x}) , \tag{1.23}$$

where $\theta^{ij}(\hat{x})$ might be any function of the generators with $\theta^{ij} = -\theta^{ji}$ and satisfying the Jacobi identity. Most commonly, the commutation relations are chosen to be either constant, linear or quadratic in the generators. In the so-called canonical case the relations are constant, i.e.

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij} = \text{const} . \tag{1.24}$$

We will discuss this case in the following Section 2. The linear or Lie-algebra case

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\lambda^{ij}_k \hat{x}^k , \tag{1.25}$$

where $\lambda^{ij}_k \in \mathbb{C}$ are the structure constants, basically has been discussed in two different settings, namely fuzzy spaces and κ-deformation. These will be briefly discussed in Sections 3.3 and 3.4, respectively. The third commonly used choice is a quadratic commutation relation,

$$\left[\hat{x}^i, \hat{x}^j\right] = \left(\frac{1}{q}\hat{R}^{ij}_{kl} - \delta^i_l \delta^j_k\right) \hat{x}^k \hat{x}^l , \tag{1.26}$$

where $\hat{R}^{ij}_{kl} \in \mathbb{C}$ is the so-called $\hat{R}$-matrix, corresponding to quantum groups. Section 3.5 will finally be devoted to this case.

Additionally, various choices of commutators $\left[\hat{x}^i, \hat{p}_j\right]$ and $[\hat{p}_i, \hat{p}_j]$ are found in the literature. In the following section, however, we start with the simplest case of constant θ^{ij} and vanishing $[\hat{p}_i, \hat{p}_j]$.

[1] An element is called self-adjoint if $x_i = x_i^*$.

2 Models with constant space-time commutator

2.1 Weyl quantization

We consider a non-commutative space where the coordinates x^μ are replaced by the Hermitian generators of a non-commutative C^*-algebra which obey the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}\,, \tag{2.1}$$

where $\theta^{\mu\nu}$ is a constant real-valued antisymmetric $D \times D$ matrix with dimension of length squared. Weyl quantization[1] provides a one-to-one correspondence between the algebra of fields on $\mathbb{R}^D$ and the algebra of according operators. (It may be thought of as an analogue of the operator-state correspondence of local QFT.) For this purpose, consider Euclidean $\mathbb{R}^D$ and fields living in an appropriate Schwartz space of functions of sufficiently rapid decrease at infinity, i.e. characterized by the condition

$$\sup_x (1+|x|^2)^{k+n_1+\ldots+n_D} |\partial_1^{n_1}\ldots\partial_D^{n_D} f(x)|^2 < \infty\,, \qquad k, n_i \in \mathbb{N}\,. \tag{2.2}$$

This implies that any function $f(x)$ may be described by its Fourier transform

$$\tilde{f}(k) = \int d^D x\, e^{-\mathrm{i}k_\mu x^\mu} f(x)\,. \tag{2.3}$$

We introduce a functions Weyl symbol by

$$\hat{\mathcal{W}}[f] := \int d^D x\, f(x)\hat{\Delta}(x)\,, \qquad \hat{\Delta}(x) = \int \frac{d^D k}{(2\pi)^D} e^{\mathrm{i}k_\mu \hat{x}^\mu} e^{-\mathrm{i}k_\mu x^\mu}\,, \tag{2.4}$$

where we have chosen the symmetric Weyl ordering prescription for operators. The operator $\hat{\Delta}(x)$ is Hermitian and describes a mixed basis for operators and fields. Hence, we may interpret the field $f(x)$ as the coordinate space representation of the Weyl operator $\hat{\mathcal{W}}[f]$. Additionally we define an anti-Hermitian linear derivation operator $\hat{\partial}_\mu$ by the commutation relations

$$\left[\hat{\partial}_\mu, \hat{x}^\nu\right] = \delta^\nu_\mu\,, \qquad \left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0\,, \tag{2.5}$$

from which it follows that

$$\begin{aligned} \left[\hat{\partial}_\mu, \hat{\Delta}(x)\right] &= -\partial_\mu \hat{\Delta}(x)\,, \qquad \left[\hat{\partial}_\mu, \hat{\mathcal{W}}[f]\right] = \hat{\mathcal{W}}[\partial_\mu f]\,, \\ e^{v^\mu \hat{\partial}_\mu} \hat{\Delta}(x) e^{-v^\mu \hat{\partial}_\mu} &= \hat{\Delta}(x+v)\,, \qquad v \in \mathbb{R}^D\,. \end{aligned} \tag{2.6}$$

[1] In fact, Weyl quantization also works when $\theta^{\mu\nu}$ is not constant.

The last relation tells us that translation generators can be represented by unitary operators $e^{v^\mu \hat{\partial}_\mu}$, and furthermore that the trace of $\hat{\Delta}(x)$ (being cyclic) must be independent of x. Choosing the normalization $\mathrm{Tr}\hat{\Delta}(x) = 1$, it follows that

$$\mathrm{Tr}\hat{\mathcal{W}}[f] = \int d^D x\, f(x)\,. \tag{2.7}$$

The products of operators $\hat{\Delta}(x)$ may be computed by using the Baker-Campbell-Hausdorff formula. In our case of constant $\theta^{\mu\nu}$ we simply have

$$e^{\mathrm{i}k_\mu \hat{x}^\mu} e^{\mathrm{i}k'_\nu \hat{x}^\nu} = e^{-\frac{\mathrm{i}}{2}k_\mu \theta^{\mu\nu} k'_\nu} e^{\mathrm{i}(k_\mu + k'_\mu)\hat{x}^\mu}\,, \tag{2.8}$$

and hence

$$\hat{\Delta}(x)\hat{\Delta}(y) = \iint \frac{d^D k}{(2\pi)^D}\frac{d^D k'}{(2\pi)^D} e^{\mathrm{i}(k_\mu + k'_\mu)\hat{x}^\mu} e^{-\frac{\mathrm{i}}{2}k_\mu \theta^{\mu\nu} k'_\nu} e^{-\mathrm{i}k_\mu x^\mu - \mathrm{i}k'_\mu y^\mu}\,. \tag{2.9}$$

Since $\mathrm{Tr} e^{\mathrm{i}k_\mu \hat{x}^\mu} = (2\pi)^D \delta^D(k_\mu)$ due to our normalization $\mathrm{Tr}\hat{\Delta}(x) = 1$, one easily derives the trace of the above expression:

$$\mathrm{Tr}\hat{\Delta}(x)\hat{\Delta}(y) = \delta^D(x-y)\,, \tag{2.10}$$

i.e. the $\hat{\Delta}(x)$ form an orthonormal set, and along with the definition of the Weyl symbol $\hat{\mathcal{W}}[f]$, it follows that the map $f(x) \mapsto \hat{\mathcal{W}}[f]$ is invertible. Hence[2]

$$f(x) = \mathrm{Tr}\left(\hat{\mathcal{W}}[f]\hat{\Delta}(x)\right)\,. \tag{2.11}$$

Having an isomorphic map between Weyl symbols and according Schwartz functions, we can in practise work with functions and a deformed product instead of with operators, provided

$$\hat{\mathcal{W}}[f]\hat{\mathcal{W}}[g] = \hat{\mathcal{W}}[f \star g]\,, \tag{2.12}$$

with $\star$ denoting a deformed product. In order to verify this and determine the explicit form of the $\star$-product, we consult (2.4) and (2.9) leading to the so-called Groenewold-Moyal $\star$-product

$$\begin{aligned} f(x) \star g(x) &= \iint \frac{d^D k}{(2\pi)^D}\frac{d^D k'}{(2\pi)^D} \tilde{f}(k)\tilde{g}(k') e^{-\frac{\mathrm{i}}{2}k_\mu \theta^{\mu\nu} k'_\nu} e^{-\mathrm{i}(k_\mu + k'_\mu)x^\mu} \\ &= f(x) e^{\frac{\mathrm{i}}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}} g(x)\,, \end{aligned} \tag{2.13}$$

where the second line is to be understood as the formal expression for a Taylor series expansion. This star product is associative but non-commutative and is defined for constant (possibly degenerate) $\theta^{\mu\nu}$. We should also mention further possibilities of writing Eqn. (2.13), namely

$$\begin{aligned} f(x) \star g(x) &= \int \frac{d^D k}{(2\pi)^D} \int d^D z f(x + \tfrac{1}{2}\theta k) g(x+z) e^{\mathrm{i}k_\mu z^\mu} \\ &= \frac{1}{\pi^D |\det\theta|} \iint d^D y\, d^D z f(x+y) g(x+z) e^{-2\mathrm{i}y^\mu \theta^{-1}_{\mu\nu} z^\nu}\,, \end{aligned} \tag{2.14}$$

[2] In fact, a function $f(x)$ obtained in this way from a quantum operator is usually called a Wigner distribution function, and therefore the map defined using $\hat{\Delta}(x)$ provides a one-to-one correspondence between Wigner fields and Weyl operators. One hence speaks of the “Weyl-Wigner correspondence”.

where the second line is only true if $\theta^{\mu\nu}$ is invertible. The last version of the star product enables us to compute the star product of two Dirac delta functions:

$$\delta^D(x) \star \delta^D(x) = \frac{1}{\pi^D |\det \theta|}\,, \tag{2.15}$$

i.e. the star product of two point sources becomes infinitely non-local. This means that very high energy processes can have important long-distance consequences.

Note the following properties:

$$\begin{aligned}
[x^\mu \stackrel{\star}{,} f(x)] &= \mathrm{i}\theta^{\mu\nu}\partial_\nu f(x)\,, \\
[f(x) \stackrel{\star}{,} g(x)] &= 2\mathrm{i} f(x) \sin\left(\tfrac{1}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right) g(x)\,, \\
\{f(x) \stackrel{\star}{,} g(x)\} &= 2 f(x) \cos\left(\tfrac{1}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right) g(x)\,,
\end{aligned} \tag{2.16}$$

as well as the extension

$$f_1(x) \star \cdots \star f_m(x) = \iiint \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_m}{(2\pi)^D}\, e^{\mathrm{i}\sum\limits_{i=1}^{m} k_i x}\, \tilde{f}_1(k_1) \cdots \tilde{f}_m(k_m) e^{-\frac{\mathrm{i}}{2}\sum\limits_{i<j}^{m} k_i \theta k_j}\,. \tag{2.17}$$

Exercise 2 *Verify Eqns.* (2.14), (2.16) *and* (2.17) *by explicit computations!*

Furthermore note that due to cyclicity of the operator trace, the integral

$$\mathrm{Tr}\left(\hat{\mathcal{W}}[f_1] \cdots \hat{\mathcal{W}}[f_m]\right) = \int d^D x f_1(x) \star \cdots \star f_m(x)\,, \tag{2.18}$$

is invariant under cyclic permutations of the f_i. Especially, one easily verifies that

$$\int d^D x f_1(x) \star f_2(x) = \int d^D x f_1(x) f_2(x)\,. \tag{2.19}$$

Finally, one also has

$$\frac{\delta}{\delta f_1(y)} \int d^D x \left(f_1 \star f_2 \star \cdots \star f_m\right)(x) = \left(f_2 \star \cdots \star f_m\right)(y)\,, \tag{2.20}$$

which follows from $\frac{\delta f_1(x)}{\delta f_1(y)} = \delta^D(x-y)$ and Eqn. (2.17).

Exercise 3 *Check relations* (2.18), (2.19) *and* (2.20)*! (Hint: Consider an inverse Fourier transformation to compute* $\frac{\delta \tilde{f}_1(k)}{\delta f_1(y)} = e^{-\mathrm{i}k_\mu y^\mu}$ *which then leads to* (2.20)*.)*

2.2 Non-commutative scalar fields and UV/IR mixing

Applying the Weyl quantization procedure of Section 2.1 to ϕ^4 theory in Euclidean $\mathbb{R}^4$, one arrives at the following action[3]:

$$\begin{aligned}
S &= \mathrm{Tr}\left(\frac{1}{2}\left[\hat{\partial}_\mu, \hat{\mathcal{W}}[\phi]\right]^2 + \frac{m^2}{2}\hat{\mathcal{W}}[\phi]^2 + \frac{\lambda}{4!}\hat{\mathcal{W}}[\phi]^4\right) \\
&= \int d^4 x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\right)\,.
\end{aligned} \tag{2.21}$$

[3] Minkowski space-time with non-commutative time is another story since time ordering needs to be redefined in that setting. We will come back to that issue in Section 2.7.

The first one to consider this action was T. Filk (1996) who derived the corresponding Feynman rules, noticing that — at least in Euclidean space — the propagator is exactly the same as in commutative space, i.e.

$$G^{\phi\phi}(k) = \frac{1}{(k^2+m^2)}\,, \tag{2.22}$$

while the vertex gains phase factors (in this case a combination of cosines) in the momenta:

$$\begin{aligned} V(k_1,k_2,k_3,k_4) = \frac{\lambda}{3}(2\pi)^4\delta^4\left(k_1+k_2+k_3+k_4\right)\Bigg[\cos\left(\frac{1}{2}\,k_1\tilde{k}_2\right)\cos\left(\frac{1}{2}\,k_3\tilde{k}_4\right) \\ + \cos\left(\frac{1}{2}\,k_1\tilde{k}_3\right)\cos\left(\frac{1}{2}\,k_2\tilde{k}_4\right) + \cos\left(\frac{1}{2}\,k_1\tilde{k}_4\right)\cos\left(\frac{1}{2}\,k_2\tilde{k}_3\right)\Bigg]\,, \end{aligned} \tag{2.23}$$

where we have introduced the short-hand notation $\tilde{k}^\mu := \theta^{\mu\nu}k_\nu$. As a consequence, new types of Feynman graphs appear: In addition to the ones known from commutative space, where no phases depending on internal loop momenta appear and which exhibit the usual UV divergences, so-called non-planar graphs come into the game which are regularized by phases depending on internal momenta. Other authors performed explicit one-loop calculations and discovered the infamous UV/IR mixing problem: Due to the phases in the non-planar graphs, their UV sector is regularized on the one hand, but on the other hand this regularization implies divergences for small external momenta instead.

For example the two point tadpole graph (in 4 dimensional Euclidean space) is approximately given by the integral

$$\Pi(\Lambda,p) = \frac{\lambda}{6}\int\frac{d^4k}{(2\pi)^4}\frac{2+\cos(k\tilde{p})}{k^2+m^2} \equiv \Pi^{\text{plan}}(\Lambda) + \Pi^{\text{n-pl}}(p)\,. \tag{2.24}$$

The planar contribution is as usual quadratically divergent in the UV cutoff Λ, i.e. $\Pi^{UV}\sim\Lambda^2$, and the non-planar part is regularized by the cosine. Using Schwinger parametrization one easily computes

$$\begin{aligned} \Pi^{\text{n-pl}} &= \frac{\lambda}{12}\int\frac{d^4k}{(2\pi)^4}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\exp\left(-\alpha(k^2+m^2)+\mathrm{i}\eta k\tilde{p}\right) \\ &= \frac{\lambda}{6(4\pi)^2}\int\limits_0^\infty\frac{d\alpha}{\alpha^2}\exp\left(-\alpha m^2-\frac{\tilde{p}^2}{4\alpha}\right) = \frac{\lambda}{24\pi^2}\sqrt{\frac{m^2}{\tilde{p}^2}}K_1\left(\sqrt{\tilde{p}^2m^2}\right) \\ &\approx \frac{\lambda}{24\pi^2}\left(\frac{1}{\tilde{p}^2}+\frac{m^2}{2}\ln|\tilde{p}^2m^2|\right)+\mathcal{O}(1)\,, \end{aligned} \tag{2.25}$$

where K_1 is the modified Bessel function and in the last line we have expanded the expression for small $\tilde{p}^2$, i.e. $\frac{1}{z}K(z)\approx\frac{1}{z^2}+\frac{1}{2}\ln z+\mathcal{O}(1)$.

This shows that the original UV divergence is not present any more, but reappears when $\tilde{p}\to 0$ (where the phase is 1) representing a new kind of infrared divergence. Since both divergences are related to one another, one speaks of "UV/IR mixing". It is this mixing which renders the action (2.21) non-renormalizable at higher loop orders. The reason is that the IR divergence cannot be absorbed by a mass redefinition, and hence a chain of such two point tadpole graphs inserted into a higher loop graph will lead to divergences of arbitrary order.

The Grosse-Wulkenhaar model.
In 2004, the first renormalizable scalar field model in non-commutative Euclidean $\mathbb{R}^4_\theta$ was introduced by H. Grosse and R. Wulkenhaar. Their trick was to add a harmonic oscillator-like term to the action (2.21), i.e.

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2}\phi \star \phi + 2\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right), \tag{2.26}$$

where $\tilde{x}_\mu := (\theta^{-1})_{\mu\nu} x^\nu$. This action cures the infamous UV/IR mixing problem and has been proved to be renormalizable to all orders in perturbation theory. This was first done by the authors employing a "matrix base", and later confirmed by V. Rivasseau *et al.* using a different tool referred to as Multiscale Analysis. Additionally, this model has another nice feature compared to the usual scalar model in commutative space-time: it has no Landau ghost[4]. Moreover, the β-function vanishes at the self-dual point $\Omega = 1$ — cp. Eqn. (2.29). Hence, this is not only the first renormalizable model on a non-commutative space, it is also an example of improved behaviour in the non-commutative setting.

Let us discuss it in some more detail: The propagator of the model is the inverse of the operator $(-\Delta + 4\Omega^2 \tilde{x}^2 + m^2)$, and is known as the Mehler kernel

$$K_M(x, y) = \int\limits_0^\infty d\alpha \, \frac{1}{4\pi^2 \omega \sinh^2 \alpha} e^{-\frac{1}{4\omega}\left(u^2 \coth \frac{\alpha}{2} + v^2 \tanh \frac{\alpha}{2}\right) - \omega m^2 \alpha}, \tag{2.27}$$

with $\omega = \frac{\theta}{2\Omega}$, $u = x - y$ and $v = x + y$.

It is also illuminating to rewrite the Grosse-Wulkenhaar (GW) action (2.26) as

$$S[\phi] = \int d^4x \left(\frac{1}{2}\phi \star [\tilde{x}_\mu \stackrel{\star}{,} [\tilde{x}^\mu \stackrel{\star}{,} \phi]] + \frac{m^2}{2}\phi \star \phi + \frac{\Omega^2}{2}\phi \star \{\tilde{x}^\mu \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} \phi\}\} + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right), \tag{2.28}$$

using the properties $[x^\mu \stackrel{\star}{,} f(x)] = \mathrm{i}\theta^{\mu\nu}\partial_\nu f(x)$ and $\{x^\mu \stackrel{\star}{,} f(x)\} = 2x^\mu f(x)$ of the star product. Written in this way, we see a symmetry between commutators and anticommutators, which in turn corresponds to a symmetry between x-space and momentum space: By exchanging $\tilde{x} \leftrightarrow p$ one can see that the action (2.26) stays form invariant:

$$S[\phi; m, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]. \tag{2.29}$$

This symmetry is called Langmann-Szabo duality. Also notice, that the Mehler kernel (2.27) shares this property. Furthermore, Eqn. (2.28) implies that the GW model may equivalently be formulated as a matrix model, i.e. where x and ϕ are matrices.

In fact, something similar was done in the original paper, namely the fields were expanded according to

$$\phi(x) = \sum_{m,n \in \mathbb{N}^{D/2}} \phi_{mn} b_{mn}(x), \tag{2.30}$$

[4]In the process of renormalizing the coupling of models such as ϕ^4 theory or QED in ordinary commutative space-time, one finds unphysical poles at very large but finite energies. This is referred to as the Landau ghost problem.

and the star products is represented by a matrix product. We shall not go into much detail of this technique, but point out another nice feature of the matrix ansatz: lines in Feynman graphs are replaced by ribbons, where the two sides represent matrix indices. This very illustrative way of drawing the Feynman graphs of the non-commutative GW model on a two dimensional Riemann surface exhibits a new way of classifying types of graphs based on their topology. One distinguishes between planar and non-planar, as well as between regular and irregular graphs[5]: In particular, one has genus $g = 0$ for *planar* and $g \geq 1$ for *non-planar* graphs. Planar graphs are then subclassified into *regular*, if their number of "broken faces" $B = 1$, and *irregular* if $B \geq 2$. The genus is related to the Euler characteristic χ and may be determined from

$$\chi = 2 - 2g = V - I + F \,, \tag{2.31}$$

where V denotes the number of vertices, I the number of internal propagators (double lines) and F is the number of "faces" (i.e. single lines) — see e.g. [4] for further details.

The scalar $1/p^2$ model.
An alternative approach to tackle the problem of UV/IR mixing was proposed by Gurau, Magnen, Rivasseau and Tanasa (2008). The main idea is to replace the GW-oscillator term by the non-local term

$$S_{\text{nloc}}[\phi] = -\int d^4x \phi(x) \star \frac{a^2}{\theta^2 \Box_x} \star \phi(x) \,, \tag{2.32}$$

where a is a dimensionless constant. The practical motivation for this is clearly to provide a counter term for the expected quadratic IR divergence in the external momentum.

The action (2.21) including the non-local addition (2.32) in momentum space reads

$$S[\phi] = \int d^4k \left[\frac{1}{2} \left(k_\mu \phi(-k) k_\mu \phi(k) + m^2 \phi^2 + a^2 \phi(-k) \frac{1}{\tilde{k}^2} \phi(k) \right) + \frac{\lambda}{4!} \mathcal{F}\left(\phi^{\star 4}\right) \right] \,, \tag{2.33}$$

where $\mathcal{F}\left(\phi^{\star 4}\right)$ denotes the Fourier transform of $\phi^{\star 4}$. Variation of the bilinear part of the action (2.33) with respect to ϕ immediately leads to the propagator

$$G(k) = \frac{1}{k^2 + m^2 + \frac{a^2}{\tilde{k}^2}} \,. \tag{2.34}$$

This Green function is the core achievement of the approach by Gurau *et al.* since it features a damping behaviour in the IR while not affecting the UV region, i.e.

$$\lim_{k \to 0} G(k) = \lim_{k \to \infty} G(k) = 0, \qquad \forall a \neq 0 \,. \tag{2.35}$$

In Multiscale Analysis this also allows the propagator to be bounded from above by a constant which is a basic ingredient leading to the renormalizability of the model. In contrast

[5] In fact, there is some confusion in the literature concerning the term "non-planar": While some authors use the topological classification, others use the term for any graph exhibiting crossing of internal lines, which is equivalent to saying their UV sector is regularized by the typical phase factors. This feature, however, is also shared by planar irregular graphs in the topological classification — an example would be the IR divergent two point tadpole graph (2.25) discussed above, which is called either "planar irregular" or "non-planar" depending on the classification scheme.

to the propagator, the vertex functional is not altered in comparison to the version (2.21) of $\phi_4^{\star 4}$ theory. The *damping* effect of the propagator (2.34) becomes obvious when considering higher loop orders. An n-fold insertion of the divergent one-loop result[6] (2.25) into a single large loop can be written as

$$\Pi^{n\ \text{np-ins.}}(p) \approx \lambda^2 \int d^4k \, \frac{e^{ik\theta p}}{\left(\tilde{k}^2\right)^n \left[k^2 + m^2 + \frac{a'^2}{\tilde{k}^2}\right]^{n+1}} \,. \tag{2.36}$$

For the model where $a = 0$, the integral of Eqn. (2.36) involves an IR divergence for $n \geq 2$, because the integrand scales as $(k^2)^{-n}$ for $k^2 \to 0$. In contrast, for the $1/p^2$ model (where $a \neq 0$), the integrand behaves like

$$\frac{1}{\left(\tilde{k}^2\right)^n \left[\frac{a'^2}{\tilde{k}^2}\right]^{n+1}} = \frac{\tilde{k}^2}{(a'^2)^{n+1}} \,, \tag{2.37}$$

i.e. it scales like $\tilde{k}^2$ independent of the order n.

2.3 Introducing gauge fields on θ-deformed spaces

A basic ingredient in the standard model of particle physics is the concept of Lie algebra valued gauge fields. In particular, the Lie groups $U(1)$, $SU(2)$ and $SU(3)$ are needed, and obviously gauge transformations need to form a closed Lie algebra, i.e.

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{-\mathrm{i}[\alpha,\beta]} \,, \qquad [\alpha,\beta] = \mathrm{i}\alpha^a\beta^b f^{abc}T^c \,, \tag{2.38}$$

where $\alpha = \alpha^a T^a$, $\beta = \beta^a T^a$, and T^a denote the generators of the Lie group in an appropriate matrix representation. However, in the non-commutative case, things are not that simple. To see the differences, we start by extending the Weyl quantisation map to matrix valued functions $\alpha(x) = \alpha^a(x)T^a$ using the tensor product between the coordinate and matrix representations:

$$\hat{\mathcal{W}}[\alpha] := \int d^D x \hat{\Delta}(x) \otimes \alpha(x) \,, \tag{2.39}$$

where $\hat{\Delta}(x)$ is defined Eqn. (2.4). We may then compute the star commutator of two Lie algebra valued functions (e.g. gauge parameters) $\alpha, \beta \in SU(N)$, and find that the result is not Lie algebra valued any more:

$$[\alpha \stackrel{\star}{,} \beta] = \frac{1}{2}\left\{\alpha^a \stackrel{\star}{,} \beta^b\right\}\left[T^a, T^b\right] + \frac{1}{2}\left[\alpha^a \stackrel{\star}{,} \beta^b\right]\left\{T^a, T^b\right\} \notin SU(N) \,. \tag{2.40}$$

The reason is that the star commutator $\left[\alpha^a \stackrel{\star}{,} \beta^b\right] \neq 0$, and as a result a term proportional to $\left\{T^a, T^b\right\}$ appears. However, the r.h.s. of Eqn. (2.40) is an element of the enveloping algebra (in this example $U(N)$). It has in fact been shown that only enveloping algebras, such as $U(N)$ or $O(N)$ and $USp(2N)$, survive the introduction of a deformed product (in the sense that commutators of algebra elements are again algebra elements), while e.g. $SU(N)$ does

[6] Note that, for the sake of simplicity, we neglect any effects due to recursive renormalization, and approximate the insertions of irregular single loops by the most divergent (quadratic) IR divergence.

not. Despite this fact, star-commutators in general do not vanish. Hence, any Groenewold-Moyal deformed gauge theory is of the non-Abelian type. In the general case, gauge fields and parameters now depend on infinitely many parameters, since the enveloping algebra on Groenewold-Moyal space is infinite dimensional. In order to emphasize this fact, we denote such algebras by $U_\star(N)$, $O_\star(N)$, $USp_\star(2N)$, ..., i.e. with subscript "$\star$". But nevertheless the parameters can be reduced to a finite number, namely the classical parameters, by the so-called Seiberg-Witten maps which we will discuss in Section 2.4.

Let us for now consider $U_\star(N)$ gauge fields $A_\mu(x) = A^a_\mu(x) T^a$ on non-commutative $\mathbb{R}^D$, i.e. let T^a be generators of $U(N)$ with $\mathrm{tr}_N(T^a T^b) = \delta^{ab}$, $a, b = 1, \ldots, N^2$, and $\left[T^a, T^b\right] = \mathrm{i} f^{abc} T^c$. We may then write a non-commutative version of the Yang-Mills action as

$$
\begin{aligned}
S_{YM} &= \frac{1}{4} \mathrm{Tr} \otimes \mathrm{tr}_N \left(\left[\hat{\partial}_\mu, \hat{\mathcal{W}}[A]_\nu\right] - \left[\hat{\partial}_\nu, \hat{\mathcal{W}}[A]_\mu\right] - \mathrm{i}g \left[\hat{\mathcal{W}}[A]_\mu, \hat{\mathcal{W}}[A]_\nu\right] \right)^2 \\
&= \frac{1}{4} \int d^D x \, \mathrm{tr}_N \left(F_{\mu\nu}(x) \star F^{\mu\nu}(x) \right) , \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu\right] .
\end{aligned} \tag{2.41}
$$

Note, that the star commutator term in the field strength is present even in the special case of $N = 1$, i.e. $U_\star(1)$ gauge fields are *non-Abelian*. The action above is invariant under the gauge transformations

$$
\begin{aligned}
\hat{\mathcal{W}}[A]_\mu &\to \hat{\mathcal{W}}[u] \hat{\mathcal{W}}[A_\mu] \hat{\mathcal{W}}[u]^\dagger - \frac{\mathrm{i}}{g} \hat{\mathcal{W}}[u] \left[\hat{\partial}_\mu, \hat{\mathcal{W}}[u]^\dagger\right] , \\
\hat{\mathcal{W}}[u] \hat{\mathcal{W}}[u]^\dagger &= \hat{\mathcal{W}}[u]^\dagger \hat{\mathcal{W}}[u] = \mathbb{1} \otimes \mathbb{1}_N ,
\end{aligned} \tag{2.42}
$$

where $\hat{\mathcal{W}}[u]$ are unitary elements of the C^*-algebra of matrix-valued Weyl operators. This implies the star gauge transformations

$$
\begin{aligned}
A_\mu(x) &\to u(x) \star A_\mu(x) \star u(x)^\dagger - \frac{\mathrm{i}}{g} u(x) \star \partial_\mu u(x)^\dagger , \\
F_{\mu\nu}(x) &\to u(x) \star F_{\mu\nu}(x) \star u(x)^\dagger , \\
u(x) \star u(x)^\dagger &= u(x)^\dagger \star u(x) = \mathbb{1}_N ,
\end{aligned} \tag{2.43}
$$

where $u(x)$ is star unitary. Notice, that the field strength $F_{\mu\nu}$ transforms covariantly even in the $U_\star(1)$ case due to the star product. Considering $u(x) = \exp\left(-\mathrm{i}g\alpha^a(x) T^a\right)$ we find the according infinitesimal gauge transformations

$$
\begin{aligned}
\delta_\alpha A_\mu(x) &= D_\mu \alpha(x) = \partial_\mu A(x) - \mathrm{i}g \left[A_\mu(x) \stackrel{\star}{,} \alpha(x)\right] , \\
\delta_\alpha F_{\mu\nu}(x) &= -\mathrm{i}g \left[F_{\mu\nu}(x) \stackrel{\star}{,} \alpha(x)\right] , \\
A_\mu(x) &= A^a_\mu(x) T^a , \qquad \alpha(x) = \alpha^a(x) T^a , \qquad F_{\mu\nu} = F^a_{\mu\nu}(x) T^a ,
\end{aligned} \tag{2.44}
$$

where formula (2.40) needs to be applied to the star commutators.

$U_\star(1)$ gauge fields.
Let us restrict ourselves to the case $N = 1$ for simplicity. Being non-Abelian, the $U_\star(1)$ group has "effective" structure constants which can easily be read off using Eqn. (2.16), namely in momentum space they are given by momentum dependent functions

$$
f(k, p, q) = 2 \sin\left(k_\mu \theta^{\mu\nu} p_\nu\right) (2\pi)^D \delta^D(p + q + k) . \tag{2.45}
$$

The effective symmetric tensors $d(p,q,k)$ take the same form with sin replaced by cos. Adapting the Faddeev-Popov technique to the non-commutative case and choosing a covariant gauge fixing, one finds the following gauge field action in θ-deformed $\mathbb{R}^4$

$$\begin{aligned} S &= \int d^4x \left(\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu} + s\left(\bar{c}\star\partial^\mu A_\mu + \tfrac{\xi}{2}\bar{c}\star b\right)\right) \\ &= \int d^4x \left(\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu} + b\star\partial^\mu A_\mu + \tfrac{\xi}{2} b\star b - \bar{c}\star\partial^\mu D_\mu c\right)\,, \end{aligned} \tag{2.46}$$

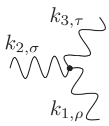
with the BRST transformations

$$\begin{aligned} &sA_\mu = \partial_\mu c - \mathrm{i}g\,[A_\mu \overset{\star}{,} c] = D_\mu c\,, && sc = \mathrm{i}gc\star c\,,\\ &s\bar{c} = b\,, && sb = 0\,,\\ &s^2\phi = 0\,, \quad \forall\phi\,. \end{aligned} \tag{2.47}$$

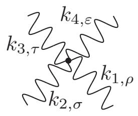
The propagators of this model in momentum space are given by

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\,, \qquad G^{c\bar{c}}(k) = -\frac{1}{k^2}\,, \tag{2.48}$$

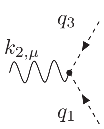
i.e. they are the same as in the according model in commutative Euclidean space. The vertices, however, are changed due to the star product: In contrast to the commutative case, one has interactions between the gauge fields and the ghosts even in the $U_\star(1)$ case due to the effective structure constants $f(k,p,q)$, $d(k,p,q)$ — see Eqn. (2.45). The vertices of the present model read



$$\begin{aligned} &= \widetilde{V}^{3A}_{\rho\sigma\tau}(k_1,k_2,k_3) \\ &= 2\mathrm{i}g(2\pi)^4\delta^4(k_1+k_2+k_3)\sin\left(\tfrac{k_1\tilde{k}_2}{2}\right)\times \\ &\quad \times\left[(k_3-k_2)_\rho\delta_{\sigma\tau} + (k_1-k_3)_\sigma\delta_{\rho\tau} + (k_2-k_1)_\tau\delta_{\rho\sigma}\right]\,, \end{aligned} \tag{2.49a}$$



$$\begin{aligned} &= \widetilde{V}^{4A}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4) \\ &= -4g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\times \\ &\quad \times\Big[(\delta_{\rho\tau}\delta_{\sigma\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{k_1\tilde{k}_2}{2}\right)\sin\left(\tfrac{k_3\tilde{k}_4}{2}\right) \\ &\quad +(\delta_{\rho\sigma}\delta_{\tau\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{k_1\tilde{k}_3}{2}\right)\sin\left(\tfrac{k_2\tilde{k}_4}{2}\right) \\ &\quad +(\delta_{\rho\sigma}\delta_{\tau\epsilon} - \delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\tfrac{k_2\tilde{k}_3}{2}\right)\sin\left(\tfrac{k_1\tilde{k}_4}{2}\right)\Big]\,, \end{aligned} \tag{2.49b}$$



$$\begin{aligned} &= \widetilde{V}^{\bar{c}Ac}_{\mu}(q_1,k_2,q_3) \\ &= -2\mathrm{i}g(2\pi)^4\delta^4(q_1+k_2+q_3)q_{1\mu}\sin\left(\tfrac{q_1\tilde{q}_3}{2}\right)\,, \end{aligned} \tag{2.49c}$$

where $\tilde{q}^\mu := \theta^{\mu\nu}q_\nu$. Similar to the scalar model (2.21), new infrared divergences appear in loop calculations due to UV/IR mixing rendering the action (2.46) non-renormalizable. At one loop level, the leading IR divergent terms read

$$\Pi^{\text{IR}}_{\mu\nu}(p) \propto \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \tag{2.50a}$$

$$\Gamma^{3A,\text{IR}}_{\mu\nu\rho}(p_1,p_2,p_3) \propto \cos\left(\frac{p_1\tilde{p}_2}{2}\right) \sum_{i=1,2,3} \frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{(\tilde{p}_i^2)^2}\,. \tag{2.50b}$$

Notice, that due to $p_\mu \tilde{p}^\mu = p_\mu \theta^{\mu\nu} p_\nu = 0$ one has $p^\mu \Pi^{\text{IR}}_{\mu\nu}(p) = 0$ which is consistent with the Ward identity $p^\mu \Pi_{\mu\nu} = 0$ following from gauge invariance. In order to repair renormalizability of the gauge model, one can try to add appropriate additional terms to the action in analogy to the scalar case. However, due to gauge symmetry, this task is not straightforward. Although several ideas in this direction exist today, a rigorous proof of renormalizability is still missing.

Covariant coordinates and induced gauge theory.
Let us reconsider a scalar field ϕ which transforms covariantly under a star gauge transformation $u(x)$ as in Eqn. (2.43), i.e.

$$\phi(x) \to u(x) \star \phi(x) \star u(x)^\dagger\,. \tag{2.51}$$

In contrast to commutative spaces, $x^\mu \star \phi(x)$ does not transform covariantly due to the star product. We can, however, introduce new "covariant" coordinates

$$\widetilde{X}_\mu := \tilde{x}_\mu + gA_\mu\,, \qquad \tilde{x}_\mu := \theta^{-1}_{\mu\nu} x^\nu\,, \tag{2.52}$$

where A_μ once more denotes a gauge field with transformation properties (2.43) in order to restore this property. It hence follows that

$$\begin{aligned} \widetilde{X}_\mu &\to u(x) \star \widetilde{X}_\mu \star u(x)^\dagger\,, \\ \widetilde{X}_\mu \phi(x) &\to u(x) \star \widetilde{X}_\mu \phi(x) \star u(x)^\dagger\,. \end{aligned} \tag{2.53}$$

under star gauge transformations. This has some interesting consequences: For one, the scalar GW-model can be straightforwardly generalized to

$$S = \int d^4x \left(\frac{1}{2}\phi \star \left[\widetilde{X}_\mu \stackrel{\star}{,} \left[\widetilde{X}^\mu \stackrel{\star}{,} \phi\right]\right] + \frac{m^2}{2}\phi\star\phi + \frac{\Omega^2}{2}\phi \star \left\{\widetilde{X}^\mu \stackrel{\star}{,} \left\{\widetilde{X}_\mu \stackrel{\star}{,} \phi\right\}\right\} + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right), \tag{2.54}$$

and the scalar field ϕ hence coupled to an external gauge field A_μ.

Furthermore, observe that the gauge field strength $F_{\mu\nu}$ may be written in terms of the covariant coordinates as a commutator

$$\mathrm{i}\left[\widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu\right] = \theta^{-1} - gF_{\mu\nu}\,. \tag{2.55}$$

Since we consider $\theta^{\mu\nu}$ to be constant in this section, we may replace

$$\int d^4x F_{\mu\nu} \star F^{\mu\nu} \quad \to \quad -\frac{1}{g^2}\int d^4x \left[\widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu\right] \star \left[\widetilde{X}^\mu \stackrel{\star}{,} \widetilde{X}^\nu\right]\,, \tag{2.56}$$

without changing the e.o.m. derived from the action.

Exercise 4 *Verify this claim!*

As in the GW case, the action (2.54) is reminiscent of a matrix model. In fact, it is believed by some authors that matrix models may provide a unified description of scalar, gauge and even fermion fields coupled to gravity. In that context, gravity is an emergent effect and the reason for UV/IR mixing — see Section 3.6 for a brief introduction.

2.4 Expanding for small θ — the Seiberg-Witten map

As one generally assumes the commutator $\theta^{\mu\nu}$ to be very small (as mentioned in the introduction perhaps even of the order of the Planck length squared), it certainly makes sense to also consider an expansion of a non-commutative theory in terms of that parameter. The star product can be written as an expansion in a formal parameter θ,

$$f \star g = f \cdot g + \sum_{n=1}^{\infty} \theta^n C_n(f, g) \,. \tag{2.57}$$

In the commutative limit $\theta \to 0$, the star product reduces to the pointwise product of functions. One may ask, if there is a similar commutative limit for the fields. The answer is given by the so-called *Seiberg-Witten map*, named after its inventors N. Seiberg and E. Witten (1999). In the simplest case it maps a non-commutative $U(1)$ gauge field A_μ to a commuting $U(1)$ Maxwell field a_μ. The existence of such a map may be motivated by the fact that a certain limit of string theory with D-branes and a B-field, can lead either to a commutative or a non-commutative effective field theory depending on the regularization scheme used. Seiberg and Witten argued that consequently there must be a local map from ordinary gauge theory to non-commutative gauge theory satisfying

$$A_\mu[a] + \delta_\Lambda A_\mu[a] = A_\mu[a + \delta_\lambda a] \,, \tag{2.58}$$

where δ_λ denotes an ordinary gauge transformation and δ_Λ a non-commutative one. The Seiberg-Witten (SW) maps are solutions of this so-called "gauge-equivalence relation". Eqn. (2.58) means that doing a gauge transformation of the non-commutative gauge field A with non-commutative gauge parameter Λ is equivalent to a gauge transformation of the commuting field a with commuting gauge parameter λ. The solutions are, however, not unique: there are some ambiguities. In fact, any Seiberg-Witten map can be obtained as the composition of a fixed SW-map and a map preserving the gauge structure of the commutative theory.

In this framework the deformation parameter $\theta^{\mu\nu}$ plays the role of a constant, unquantized and external field. In this way, a θ-expanded deformed non-commutative Maxwell theory can be obtained where the photon receives a self-interaction via the background field $\theta^{\mu\nu}$.

The solution to this question for Abelian gauge groups, as given by Seiberg and Witten, reads

$$\begin{aligned}
A_\mu[a] &= a_\mu + \frac{g\theta^{\sigma\tau}}{2} \left(a_\tau \partial_\sigma a_\mu + f_{\sigma\mu} a_\tau\right) + \mathcal{O}(\theta^2) \,, \\
\Phi[\phi, a] &= \phi + \frac{g\theta^{\mu\nu}}{2} a_\nu \partial_\mu \phi + \mathcal{O}(\theta^2) \,, \\
\Lambda[\lambda, a] &= \lambda + \frac{g\theta^{\mu\nu}}{2} a_\nu \partial_\mu \lambda + \mathcal{O}(\theta^2) \,,
\end{aligned} \tag{2.59}$$

where $f_{\mu\nu}$ is the field strength tensor associated to a_μ and Φ/ϕ denote scalar fields. Generalizing to non-Abelian gauge fields a_μ on the commutative side, we start by expanding Λ in terms of θ,

$$\Lambda[A] = \Lambda^0 + \Lambda^1[a] + \Lambda^2[a] + \mathcal{O}(\theta^3)\,, \tag{2.60}$$

where Λ^n is $\mathcal{O}(\theta^n)$. Solving order by order, we arrive at

$$\begin{aligned} 0^{\text{th}}\text{ order}: \ \Lambda^0 &= \lambda\,, \\ 1^{\text{st}}\text{ order}: \ \Lambda^1 &= \frac{g\theta^{\mu\nu}}{4}\left\{\partial_\mu\lambda, a_\nu\right\}\,. \end{aligned} \tag{2.61}$$

For scalar fields Φ the condition

$$\delta_\lambda\Phi[a] = \delta_\Lambda\Phi[a] = \mathrm{i}g\Lambda[a] \star \Phi[a] \tag{2.62}$$

has to be satisfied. In other words, the ordinary gauge transformation induces a non-commutative gauge transformation. We expand the fields in terms of the non-commutativity

$$\Phi = \Phi^0 + \Phi^1[a] + \Phi^2[a] + \dots\,, \tag{2.63}$$

and solve Eqn. (2.62) order by order in θ. In first order, we have to find a solution to

$$\delta_\lambda\Phi^1[a] = \mathrm{i}g\lambda\Phi^1 + \mathrm{i}g\Lambda^1\phi - \frac{g\theta^{\mu\nu}}{2}\partial_\mu\lambda\partial_\nu\phi\,. \tag{2.64}$$

It is given by

$$\begin{aligned} 0^{\text{th}}\text{ order}: \ \Phi^0 &= \phi\,, \\ 1^{\text{st}}\text{ order}: \ \Phi^1 &= -\frac{g\theta^{\mu\nu}}{2}a_\mu\partial_\nu\phi + \frac{\mathrm{i}g\theta^{\mu\nu}}{4}a_\mu a_\nu\phi\,. \end{aligned} \tag{2.65}$$

The gauge fields A_μ have to satisfy

$$\delta_\lambda A_\mu[a] = \partial_\mu\Lambda[a] - \mathrm{i}g\left[\Lambda_\mu[a] \stackrel{\star}{,} \Lambda[a]\right]\,, \tag{2.66}$$

(cf. the gauge equivalence relation Eqn. (2.58)). Using the expansion

$$A_\mu[a] = A^0_\mu + A^1_\mu[a] + A^2_\mu[a] + \dots\,, \tag{2.67}$$

and solving (2.66) order by order, we end up with

$$\begin{aligned} 0^{\text{th}}\text{ order}: \ A^0_\mu &= a_\mu\,, \\ 1^{\text{st}}\text{ order}: \ A^1_\mu &= -\frac{g\theta^{\tau\nu}}{4}\left\{a_\tau, \partial_\nu a_\mu + f_{\nu\mu}\right\}\,, \end{aligned} \tag{2.68}$$

where $f_{\nu\mu} = \partial_\nu a_\mu - \partial_\mu a_\nu - \mathrm{i}g\left[a_\nu, a_\mu\right]$. Similarly, we have for the field strength $F_{\mu\nu}$

$$\begin{aligned} \delta_\lambda F_{\mu\nu} &= \mathrm{i}g\left[\Lambda, F_{\mu\nu}\right] \\ \text{and } F_{\mu\nu} &= f_{\mu\nu} + \frac{g\theta^{\sigma\tau}}{2}\left\{f_{\mu\sigma}, f_{\nu\tau}\right\} - \frac{g}{4}\theta^{\sigma\tau}\left\{a_\sigma, (\partial_\tau + D_\tau)f_{\mu\nu}\right\}\,, \end{aligned} \tag{2.69}$$

where $D_\mu f_{\tau\nu} = \partial_\mu f_{\tau\nu} - \mathrm{i}g\left[a_\mu, f_{\tau\nu}\right]$.

It should be pointed out that gauge field theories formulated via the Seiberg-Witten map are manifestly IR *finite* in the sense of UV/IR mixing. Only the usual UV divergences are present.

Exercise 5 *Explicitly compute the Seiberg-Witten map to first order in θ, first for Abelian a_μ verifying* (2.59)*, then generalize the results for non-Abelian a_μ and verify* (2.61)*,* (2.65) *and* (2.68)*!*

The $U_\star(1)$ gauge field action (2.46) in terms of the Seiberg-Witten expansion reads

$$S_{\text{inv}} = \int d^4x \left(\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{g}{8} \theta^{\alpha\beta} f_{\alpha\beta} f_{\mu\nu} f^{\mu\nu} + \frac{g}{2} \theta^{\alpha\beta} f_{\mu\alpha} f_{\nu\beta} f^{\mu\nu} + \mathcal{O}(\theta^2) \right) , \tag{2.70}$$

which is invariant under the usual Abelian gauge transformation δ_λ. Concerning the gauge fixing and ghost sector, there seem to be two fundamentally different ways of introducing ghosts into the theory: before or after performing the Seiberg-Witten map. While the latter with Landau gauge fixing leads to the usual terms

$$S_{\text{gf}} = \int d^4x s \left(\bar{c} \partial_\mu a^\mu + \frac{\xi}{2} \bar{c} b \right) , \tag{2.71}$$

with the BRST transformations

$$s a_\mu = \partial_\mu c \,, \qquad sc = 0 \,, \qquad s\bar{c} = b \,, \qquad sb = 0 \,, \tag{2.72}$$

performing the SW-map after introducing ghosts into the action according to (2.46), leads to

$$S_{\text{gf}} = \int d^4x \left(b(\partial a) - \bar{c} \Box c - g\theta^{\alpha\beta} \left(\partial^\mu \bar{c} \partial_\alpha c \partial_\beta a_\mu - \frac{1}{2} \Box \bar{c} a_\alpha \partial_\beta c - \frac{1}{2} \partial^\mu b a_\alpha (\partial_\beta a_\mu + f_{\beta\mu}) \right) \right) , \tag{2.73}$$

due to the SW-expansion[7] of the non-commutative fields $\hat{c}, \hat{\bar{c}}, \hat{b}$:

$$\hat{c} = c + \frac{g}{2} \theta^{\mu\nu} a_\nu \partial_\mu c + \mathcal{O}(\theta^2) \,, \qquad \hat{\bar{c}} = \bar{c} \,, \qquad \hat{b} = b \,. \tag{2.74}$$

Once more, the action (2.73) is invariant under the BRST transformations (2.72).

Although the pure gauge sector (2.70) is renormalizable, Seiberg-Witten expanded theories become *non-renormalizable* if one also adds fermions to the pure gauge sector as was proven by R. Wulkenhaar in 2001. Nevertheless, such theories may be studied as describing non-commutative corrections to commutative models, i.e. they are regarded as effective theories.

In particular, one may study a non-commutative version of the standard model of particle physics in terms of a Seiberg-Witten expansion. This gives rise to new couplings and decay modes, which might be forbidden or highly suppressed in the commutative Standard Model. For example, there is a new coupling of photons to neutral particles, and the decay $Z \to \gamma\gamma$. By studying such processes one can obtain bounds on the non-commutativity scale. The SW map has also been applied to astrophysical scenarios, e.g. bounds for the non-commutative scale can be derived from estimates for the induced energy loss in stars and from comparison of Dirac/Majorana neutrino dipole moments. Furthermore, also big bang nucleosynthesis may be used in order to constrain the scale of non-commutative effects.

Finally, it should be mentioned, that Seiberg-Witten maps can also be constructed for non-constant $\theta^{\mu\nu}$. In particular, this has been done for κ-deformed spaces (which we will introduce in Section 3.4) and for certain classes of q-deformed spaces (which are introduced in Section 3.5).

[7]Note, that we merely need to substitute $\Lambda \to \hat{c}$, $\lambda \to c$ in (2.59).

2.5 Physical applications: the Quantum Hall effect

2.5.1 Non-commutative Chern-Simons theory

We have mentioned in the introduction, that the quantum Hall effect may be described by a non-commutative version of the Chern-Simons (CS) action. In order to make the connection, we start by introducing the CS model on θ-deformed space:

As in the previous sections, we may generalize the action of Chern-Simons theory using Weyl operators. Remember that CS-theory is defined in 3 dimensions[8], and in the non-commutative case reads

$$\begin{aligned} S_{\rm CS} &= -\frac{1}{2}{\rm Tr}\otimes{\rm tr}_N\,\epsilon^{\mu\nu\rho}\left(\hat{\mathcal{W}}[A]_\mu\left[\hat\partial_\nu,\hat{\mathcal{W}}[A]_\rho\right]-\frac{2{\rm i}g}{3}\hat{\mathcal{W}}[A]_\mu\hat{\mathcal{W}}[A]_\nu\hat{\mathcal{W}}[A]_\rho\right) \\ &= -\frac{1}{2}{\rm tr}_N\int\!d^3x\,\epsilon^{\mu\nu\rho}\left(A_\mu\star\partial_\nu A_\rho-\frac{2{\rm i}g}{3}A_\mu\star A_\nu\star A_\rho\right)\,. \end{aligned} \tag{2.75}$$

It is independent of the metric and therefore a topological model. The CS action is invariant under the gauge transformations $\delta_\lambda A_\mu = D_\mu\lambda$. Furthermore, the equations of motion tell us that

$$\epsilon^{\mu\nu\rho}F_{\nu\rho}=0\,,\qquad\text{with}\quad F_{\nu\rho}=\partial_\nu A_\rho-\partial_\rho A_\nu-{\rm i}g\left[A_\nu\stackrel{\star}{,}A_\rho\right]\,, \tag{2.76}$$

from which it follows that the field strength should vanish, i.e. $F_{\mu\nu}=0$.

As in the Yang-Mills case, the gauge symmetry needs to be fixed, and for this purpose we choose a Landau gauge fixing ($\xi = 0$ in our previous notation), i.e.

$$S_{\rm gf}=s\int\!d^3x\,{\rm tr}_N\left(\bar c\star\partial_\mu A^\mu\right)=\int\!d^3x\,{\rm tr}_N\left(b\star\partial_\mu A^\mu-\bar c\star\partial^\mu D_\mu c\right)\,. \tag{2.77}$$

The complete action $S_{\rm CS}+S_{\rm gf}$ is invariant under the BRST transformations (2.47) — but due to $S_{\rm gf}$ depends on the metric, which for now we choose to be Euclidean.

It turns out, that with such a Landau gauge fixing the complete action exhibits an additional symmetry which is characterized by the transformations

$$\begin{aligned} \delta_\nu A_\rho &= \epsilon_{\mu\nu\rho}\partial^\mu\bar c\,, & \delta_\nu c &= -A_\nu\,, \\ \delta_\nu b &= -\partial_\nu\bar c\,, & \delta_\nu\bar c &= 0\,. \end{aligned} \tag{2.78}$$

For obvious reasons it is called the "linear vector supersymmetry". In some sense, it may be viewed as an "inverse" of the BRST transformations. Both symmetries together form an algebra which closes on-shell on space-time translations:

$$\begin{aligned} \{s,s\} &= 0\,, & \{\delta_\mu,\delta_\nu\} &= 0\,, \\ \{\delta_\mu,s\} &= -\partial_\mu+\text{e.o.m.} \end{aligned} \tag{2.79}$$

Exercise 6 *Check invariance of the gauge fixed Chern-Simons action under the transformations* (2.47) *and* (2.78)*, as well as the algebra* (2.79)*!*

In general, topological theories do not exhibit propagating degrees of freedom. This means, there should be no loop corrections to the propagators. It may perhaps not be clear that this should hold also in non-commutative space, but explicit one-loop calculations in fact confirm this claim.

[8] We restrict ourselves to flat space in this section.

2.5.2 Fuzzy fluids

Our aim is to describe the quantum Hall effect, and since we are going to describe the electrons in terms of a fluid, we need to discuss some related properties.

There are basically two equivalent descriptions of fluids:

1. In terms of the coordinates $X^i(x,t)$ (with $i=1,2$) of the particles comprising the fluid in the Lagrange description, where the x^i "labeling" the particles are called co-moving coordinates,

2. and in terms of density $\rho(r,t)$ and velocity fields $v^i(r,t)$ at each point in space r^i in the Euler description.

In order to make contact with the quantum Hall effect, we work in two spacial dimensions, and note that in the Lagrange description all fluid quantities must be invariant under particle relabelling if the density ρ_0 stays invariant. In the present case, this symmetry corresponds to area-preserving diffeomorphisms of the variables x^i. Infinitesimal transformations may be written as

$$\delta x^i = \epsilon^{ij}\frac{\partial f(x)}{\partial x^j}\,, \tag{2.80}$$

for which the area preserving condition $\det\left(\frac{\partial x^i+\delta x^i}{\partial x^j}\right)=1 \;\Leftrightarrow\; \frac{\partial \delta x^i}{\partial x^j}=0$ is obviously fulfilled. Introducing Poisson brackets $\{x^i,x^j\}_{\mathrm{PB}}=\theta\epsilon^{ij}$ with some constant θ, we can rewrite (2.80) as $\delta x^i=\{x^i,f\}_{\mathrm{PB}}$ and hence in terms of the fundamental Lagrange fluid variables as

$$\delta X^i = \partial_j X^i \delta x^j = \{X^i, f(x)\}_{\mathrm{PB}}\,. \tag{2.81}$$

This looks like the classical approximation of a gauge transformation of the covariant non-commutative coordinates X^i of Eqn. (2.52) in Section 2.3. The analogy becomes even more striking upon introducing a function $\mathcal{A}^i(x,t)$ measuring the deviation X^i-x^i, i.e.

$$X^i = x^i + \mathcal{A}^i(x,t) = x^i + \theta\epsilon^{ij}A_j(x,t)\,. \tag{2.82}$$

This "gauge field" A_i transforms as $\delta A_i=\partial_i f+\{A_i,f\}_{\mathrm{PB}}$. The "duals" of the X^i, denoted by $\widetilde{X}_i=\frac{1}{\theta}\epsilon_{ij}X^j=\tilde{x}_i+A_i$ obviously correspond to covariant derivatives leading to the "field strength"

$$\begin{aligned}\hat{F}_{ij} &= \{\widetilde{X}_i,\widetilde{X}_j\}_{\mathrm{PB}} = \frac{1}{\theta}\epsilon_{ij}+F_{ij}\,,\\ F_{ij} &= \partial_i A_j - \partial_j A_i + \{A_i,A_j\}_{\mathrm{PB}}\,.\end{aligned} \tag{2.83}$$

The fluid density ρ in this notation computes to

$$\begin{aligned}\rho(r,t) &= \rho_0\int dx\,\delta(X(x,t)-r)\,,\\ \Rightarrow\;\frac{\rho_0}{\rho} &= \det\left(\frac{\partial X^i(x,t)}{\partial x^j}\right) = \frac{1}{\theta}\{X^1,X^2\}_{\mathrm{PB}}\,,\end{aligned} \tag{2.84}$$

leading to the relation

$$\hat{F}_{ij} = \frac{\rho_0}{\rho}\epsilon_{ij}\,, \tag{2.85}$$

i.e. the field strength (which in 2-dimensional space has only one non-vanishing component) is related to the fluid density.

So far, we have only considered time-independent gauge transformations. In order to include time-dependent transformations, one has to introduce a temporal gauge field A_0 which transforms as

$$\delta A_0 = \dot{f} + \{A_0, f\}_{\text{PB}} \,, \tag{2.86}$$

where $f(x,t)$ is now time-dependent. More precisely, in order to render time-derivatives $\dot{X}(x,t)$ covariant with respect to gauge transformations, we must replace them by covariant time-derivatives

$$D_0 X^i = \dot{X}^i + \{A_0, X^i\}_{\text{PB}} \,, \tag{2.87}$$

in the Lagrangian. As this would change the dynamics of the system, a temporal gauge fixing $A_0 = 0$ is required.

The transition to non-commutative fluids ("fuzzy fluids") is achieved the same way as the transition from classical to quantum mechanics, i.e. by replacing the Poisson brackets with commutators:

$$\{\cdot,\cdot\}_{\text{PB}} \;\rightarrow\; -\mathrm{i}\,[\cdot,\cdot] \,, \qquad\qquad \left[x^i, x^j\right] = \mathrm{i}\theta\epsilon^{ij} \,, \tag{2.88}$$

so that the co-moving coordinates are promoted to a non-commutative plane for which we may consider Weyl quantization as discussed in Section 2.1. This then obviously leads to a non-commutative gauge theory (with commuting time). In choosing $2\pi\theta = 1/\rho_0$, summation over the particles becomes the trace over the representation space, i.e.

$$\sum_{\text{particles}} = \rho_0 \int dx \rightarrow 2\pi\theta\rho_0 \mathrm{Tr} = \mathrm{Tr} \,. \tag{2.89}$$

Note, that the fluid density $\rho(r,t)$ is still an ordinary function, as can be seen from the non-commutative version of (2.84), i.e.

$$\rho(r,t) = \mathrm{Tr}\delta(X(x,t) - r) \,. \tag{2.90}$$

The same is true for the particle current

$$j^i = \rho v^i = \rho_0 \int dx \dot{X}^i \delta(X - r) \,, \tag{2.91}$$

where v^i is the Euler velocity. Density and current still satisfy the continuity equation, which means that our fuzzy fluid still has an Euler description in terms of ordinary commuting particle density and current. Hence, a Seiberg-Witten map must exist between the two descriptions. More precisely, since in 2+1 dimensions the conserved current can be written in terms of its dual two-form $J_{\mu\nu} = \epsilon_{\mu\nu\rho} j^\rho$ with $j^0 = \rho$, which then satisfies the Bianchi identity, this J can be considered as an Abelian field strength allowing the definition of an Abelian commutative gauge field a_μ.

2.5.3 Quantum Hall fluid as non-commutative Chern-Simons model

In Section 1.2 we have discussed the Landau problem and its relation to the quantum Hall effect. We have seen, that projection onto the lowest Landau level (i.e. neglecting the kinetic term for large magnetic field) leads to non-commutative space coordinates in the plane perpendicular to the magnetic field. We now consider a large number N of electrons on the plane in the lowest Landau level. Comparing (1.11) with (2.88) exhibits the identification $\theta = 1/(eB)$.

The according action including the temporal gauge field A_0 to make it gauge invariant reads

$$\begin{aligned} S &= \int dt \frac{eB}{2} \mathrm{Tr}\left(\epsilon_{ij} D_0 X^i X^j + 2\theta A_0\right) \\ &= \int dt \frac{eB}{2} \mathrm{Tr}\left(\epsilon_{ij}\big(\dot{X}^i - \mathrm{i}\left[A_0, X^i\right]\big) X^j + 2\theta A_0\right) , \end{aligned} \tag{2.92}$$

where the last term had to be added so that the e.o.m. for A_0 leads to the constraint

$$\left[X^1, X^2\right] = \mathrm{i}\theta\,. \tag{2.93}$$

rather than zero. Furthermore, we introduce the *filling fraction* ν as the fraction

$$\nu = \frac{\rho_0}{\rho_{LLL}} = \frac{1}{\theta eB}\,, \tag{2.94}$$

where $\rho_{LLL} = eB/(2\pi)$ is the lowest Landau level density and ρ_0 was defined by (2.89) above. Notice, that the action (2.92) is nothing else than the Chern-Simons action in 2+1 dimensions (with commuting time).

Exercise 7 *Show that the action* (2.92) *may be written in a form similar to* (2.75) *upon inserting* (2.82) *and using* (2.88)*!*

2.6 The fuzzy torus

A further interesting case to study is the replacement of the flat D-dimensional Groenewold-Moyal space by a D-dimensional "fuzzy" torus $\mathbb{T}^D_\theta$. This requires only some small changes compared to the situation in Section 2.1, e.g. one needs to impose a periodicity condition such as

$$x^\mu \sim x^\mu + \Sigma^\mu_a\,, \qquad\qquad a = 1, \ldots, D\,. \tag{2.95}$$

When Σ^μ_a is not proportional to δ^μ_a, one has a tilted torus. This periodicity implies that the corresponding Fourier momenta are quantized according to

$$k_\mu = 2\pi(\Sigma^{-1})^a_\mu n_a\,, \tag{2.96}$$

with $n_a \in \mathbb{Z}$. Furthermore, one considers a Weyl basis of unitary operators

$$\hat{Z}^a = e^{2\pi\mathrm{i}(\Sigma^{-1})^a_\mu \hat{x}^\mu}\,. \tag{2.97}$$

These then generate the algebra

$$\hat{Z}^a \hat{Z}^b = e^{-2\pi \mathrm{i}\Theta^{ab}} \hat{Z}^b \hat{Z}^a \,,$$
$$\Theta^{ab} := 2\pi (\Sigma^{-1})^a_\mu \theta^{\mu\nu} (\Sigma^{-1})^b_\nu \,, \tag{2.98}$$

which replaces (2.1). Functions of $\mathbb{T}^D_\theta$ can be Fourier expanded as

$$f(x) = \sum_{\vec{m}\in\mathbb{Z}^D} f_{\vec{m}} e^{2\pi\mathrm{i}(\Sigma^{-1})^a_\mu m_a x^\mu} \,, \tag{2.99}$$

and Weyl quantization takes the form

$$\hat{\mathcal{W}}[f] := \int d^D x\, f(x) \hat{\Delta}(x) \,,$$
$$\hat{\Delta}(x) = \frac{1}{\det \Sigma} \sum_{\vec{m}\in\mathbb{Z}^D} \prod_{a=1}^{D} (\hat{Z}^a)^{m_a} \prod_{a<b} e^{-\pi \mathrm{i} m_a \Theta^{ab} m_b} e^{-2\pi\mathrm{i}(\Sigma^{-1})^a_\mu m_a x^\mu} \,. \tag{2.100}$$

Note that $\hat{\Delta}(x)$ shares the periodicity property (2.95). Finally, we may define a derivation on the non-commutative torus as

$$\left[\hat{\partial}_\mu, \hat{Z}^a\right] = 2\pi\mathrm{i}(\Sigma^{-1})^a_\mu \hat{Z}^a \,, \tag{2.101}$$

leading to the property

$$\left[\hat{\partial}_\mu, \hat{\Delta}(x)\right] = -\partial_\mu \hat{\Delta}(x) \,, \tag{2.102}$$

(cp. (2.6)).

2.7 Time ordering in non-commutative QFTs

Throughout the previous sections we have either considered Euclidean spaces, or kept time commutative, i.e. $\theta^{0\mu} = 0$. The difficulty with handling $\theta^{0\mu} \neq 0$ lies in the fact that, due to the star products, the interaction part of the Lagrangian depends on infinitely many time derivatives acting on the fields. Furthermore, unitarity of the S matrix appears to be violated when time-ordering is not adapted to the non-commutative setting. In order to pin down the problem, let us briefly review the definition of the S matrix in commutative space-time. It is given by

$$S = T \exp\left(\mathrm{i} \int d^4 x \mathcal{L}_{\mathrm{int}}(x)\right) \,, \tag{2.103}$$

where T is the time ordering operator, $\mathcal{L}_{\mathrm{int}}$ denotes the interaction part of the Lagrangian, and the S matrix is unitary by construction. If we consider a scalar field theory with an interaction of type $\mathcal{L}(x) = -\phi_{\mathrm{in}}(x) j(x)$ where ϕ_{in} denotes an incoming free scalar field and $j(x)$ is some external source, the expression (2.103) may be rewritten as

$$S = \exp\left(-\mathrm{i} \int d^4 x \phi_{\mathrm{in}}(x) j(x)\right) \exp\left(-\frac{\mathrm{i}}{2} \int d^4 x d^4 x' j(x) G_{\mathrm{ret}}(x - x') j(x')\right) \,, \tag{2.104}$$

where $G_{\rm ret}(x-x')$ is the (retarded) propagator. Obviously, the S-matrix as defined above is unitary by definition if the correct time-ordering is chosen. Hence, any non-unitary result can only mean, that the naïve approach to time-ordering in the non-commutative setting is wrong.

A possible solution has been proposed by S. Doplicher *et al.* (1994) and further developed for non-commutative scalar ϕ^4 theory by several authors. It is termed "interaction point time ordered perturbation theory" (IPTOPT) and is based on the following idea: Consider the Gell-Mann–Low formula applied to the field operators ϕ of a scalar ϕ^4 theory

$$\langle 0|T\{\phi_H(x_1)\ldots\phi_H(x_n)\}|0\rangle = \sum_{m=0}^{\infty}\frac{(-\mathrm{i})^m}{m!}\int\limits_{-\infty}^{\infty} dt_1 \int\limits_{-\infty}^{\infty} dt_2 \ldots \int\limits_{-\infty}^{\infty} dt_m \times \\ \times \langle 0|T\{\phi_I(x_1)\ldots\phi_I(x_n)V(t_1)\ldots V(t_m)\}|0\rangle\,. \tag{2.105}$$

The subscripts H and I denote the Heisenberg picture and the interaction picture, respectively. V is the interaction part of the Hamiltonian

$$V(z^0) = \int d^3z \frac{\lambda}{4!}\phi(z)\star\phi(z)\star\phi(z)\star\phi(z)\,. \tag{2.106}$$

The idea is that the time-ordering operator T acts on the time components of the x_i and on the so-called *time stamps* $t_1,\ldots,t_m$. For example, considering the interaction (2.106) with an alternative representation — cf. (2.14) — for the star products

$$V(z^0) = \frac{\lambda}{4!}\prod_{i=1}^{3}\int\frac{d^4s_i d^4l_i}{(2\pi)^4}e^{\mathrm{i}s_il_i}\phi(z-\tfrac{\theta}{2}l_1)\phi(z+s_1-\tfrac{\theta}{2}l_2)\phi(z+s_1+s_2-\tfrac{\theta}{2}l_3)\phi(z+s_1+s_2+s_3)\,, \tag{2.107}$$

the time ordering only affects z^0 and no other time components (like e.g. l_i^0 etc.). From the expression above, one furthermore notices that the interaction in non-commutative spaces is "smeared out", i.e. it is not located at one specific point in space-time as illustrated in Figure 2.1.

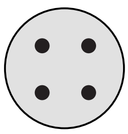


Figure 2.1: Vertices in non-commutative $\phi^{\star 4}$ theory (and others) are not located at one space-time point, but "smeared" over a region whose size is determined by θ.

This leads to modified Feynman rules, i.e. the propagator, being the time-ordered product of two fields in the tree approximation, is modified since time-ordering is modified. For example, the propagator of ϕ^4 theory

$$G(x,x') = \int\frac{d^4k}{(2\pi)^4}\frac{e^{\mathrm{i}k(x-x')}}{k^2+m^2-\mathrm{i}\epsilon}\,, \tag{2.108}$$

is generalized to the so-called *contractor*

$$G_C(x,t;x',t') = \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left[\mathrm{i}k(x-x') + \mathrm{i}k^0(x^0 - t - (x'^0 - t'))\right]}{k^2 + m^2 - \mathrm{i}\epsilon} \times$$
$$\times \left[\cos\left(\omega_k(x^0 - t - (x'^0 - t'))\right) - \frac{\mathrm{i}k^0}{\omega_k} \sin\left(\omega_k(x^0 - t - (x'^0 - t'))\right)\right], \quad (2.109)$$

which for $x^0 = t$ and $x'^0 = t'$ (being the case when $\theta^{0\mu} = 0$) reduces to (2.108). It is computed by replacing $\Theta(x_0 - x_0')$ with $\Theta(t - t')$ in the expression for the Feynman propagator, i.e.

$$G_F(x,x') = \Theta(x_0 - x_0')\left[\phi^+(x), \phi^-(x')\right] + \Theta(x_0' - x_0)\left[\phi^+(x'), \phi^-(x)\right], \quad (2.110)$$

becomes

$$G_C(x,x') = \Theta(t - t')\left[\phi^+(x), \phi^-(x')\right] + \Theta(t' - t)\left[\phi^+(x'), \phi^-(x)\right], \quad (2.111)$$

which then leads to (2.109).

Exercise 8 *Verify the expression for the contractor of* $\phi^{\star 4}$ *theory* (2.109) *by explicit computation! Remember that*

$$\left[\phi^+(x), \phi^-(x')\right] = \int \frac{d^3p}{2\omega_p(2\pi)^3} e^{p(x-x')|_{p_0 = \omega_p}},$$

and use an appropriate integral representation for the step function θ*.*

Due to these generalizations, perturbation theory becomes somewhat more involved than in the commutative case. Furthermore, many open questions (e.g. concerning renormalizability, UV/IR mixing, etc.) remain.

3 Non-canonical deformations

In the previous sections, we have thoroughly discussed gauge theories formulated on non-commutative spaces with constant $\theta^{\mu\nu}$, i.e. Groenewold-Moyal spaces. The following shall therefore give a brief overview over other approaches, such as x-dependent $\theta^{\mu\nu}$. The topics we will cover are twisted gauge theories, after which we will proceed to the case of linear dependence on x, i.e. fuzzy and κ-deformed spaces, and finally review approaches with the most general x dependence of the commutator, such as quantum groups and matrix models.

3.1 Hopf algebras

In the following sections, techniques known from Hopf algebras will be needed, and we therefore review their basic properties:

A Hopf algebra (named after Heinz Hopf) $\mathcal{A}$, denoted by $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$, consists of an associative algebra $(\mathcal{A}, m, \eta)$ with a compatible co-algebra structure, given by the structure maps Δ, ϵ and S. In detail, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication and η the unit map $\eta : \mathbb{C} \to \mathcal{A}\,,\ c \mapsto c\mathbb{1}_{\mathcal{A}}$, where $\mathbb{1}_{\mathcal{A}} \in \mathcal{A}$ is the unit element. The multiplication is associative. The structure maps of the co-algebra are by definition dual to m and η:

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}\,, \qquad\qquad \epsilon : \mathcal{A} \longrightarrow \mathbb{C}\,. \tag{3.1}$$

The co-product Δ satisfies the co-associativity rule

$$(\mathbb{1} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{1}) \circ \Delta\,, \tag{3.2}$$

and for the co-unit ϵ we have a similar defining relation

$$(\epsilon \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \epsilon) \circ \Delta\,. \tag{3.3}$$

The antipode ("inverse") S is defined via the relation

$$m \circ (S \otimes \mathbb{1}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathbb{1} \otimes S) \circ \Delta\,. \tag{3.4}$$

Compatibility between algebra and co-algebra structures means that the co-product Δ and the co-unit ϵ are algebra homomorphisms, i.e.

$$\Delta(ab) = \Delta(a)\Delta(b)\,, \qquad\qquad \epsilon(ab) = \epsilon(a)\epsilon(b)\,, \tag{3.5}$$

with $a, b \in \mathcal{A}$.

Finally, note that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{A}\otimes\mathcal{A} \arrow[rr, "S\otimes\mathbb{1}"] & & \mathcal{A}\otimes\mathcal{A} \arrow[dr, "m"] & \\
\mathcal{A} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{C} \arrow[rr, "\eta"] & & \mathcal{A} \\
& \mathcal{A}\otimes\mathcal{A} \arrow[rr, "\mathbb{1}\otimes S"'] & & \mathcal{A}\otimes\mathcal{A} \arrow[ur, "m"'] &
\end{tikzcd}$$

3.2 Twisted gauge theories

In the so-called "twisted approach", the main idea is to also deform the Leibniz rule (in addition to the pointwise product) by using Hopf algebra techniques (see Section 3.1). Following J. Wess *et al.*, consider first the undeformed (i.e. commutative) case: We define a pointwise product as

$$m\{f \otimes g\} = f \cdot g\,, \tag{3.6}$$

and the infinitesimal gauge transformation of a scalar field ϕ as

$$\delta_\alpha \phi(x) = \mathrm{i}\alpha(x)\phi(x)\,, \tag{3.7}$$

where $\alpha(x)$ is Lie algebra valued (see Section 2.3). The co-multiplication $\Delta(\alpha)$, an essential ingredient for a Hopf algebra, is defined by

$$\Delta(\alpha) = \alpha \otimes \mathbb{1} + \mathbb{1} \otimes \alpha\,, \tag{3.8}$$

and allows us to write the Leibniz rule for the gauge transformation of a product of fields in the language of Hopf algebras as

$$\begin{aligned}\delta_\alpha(\phi_1 \cdot \phi_2) &= (\delta_\alpha\phi_1)\phi_2 + \phi_1(\delta_\alpha\phi_2) \\ &= m\{\Delta(\alpha)\phi_1 \otimes \phi_2\}\,.\end{aligned} \tag{3.9}$$

In the *deformed* case, on the other hand, one has to replace the pointwise product (3.6) with a deformed version, which in the simplest case could be the Groenewold-Moyal product of Section 2.1, i.e. in the Hopf algebra language

$$m_\star\{f \otimes g\} = m\{e^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g\}\,. \tag{3.10}$$

The non-commutative gauge transformation $\delta^\star_\alpha$ on a single field is defined as

$$\delta^\star_\alpha \phi = \mathrm{i}\alpha \cdot \phi\,, \tag{3.11}$$

as in the commutative case. Furthermore, one considers a deformed — or "twisted" — co-product

$$\begin{aligned}\Delta_{\mathcal{F}}(\alpha) &= \mathcal{F}(\alpha \otimes \mathbb{1} + \mathbb{1} \otimes \alpha)\mathcal{F}^{-1}\,, \\ \mathcal{F} &= e^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}\,,\end{aligned} \tag{3.12}$$

where $\mathcal{F}$ denotes a "twist operator" that has all the properties to define a Hopf algebra with (3.12) as a co-multiplication. Hence, we may write a Groenewold-Moyal deformed version of the Leibniz rule (3.9) as

$$\begin{aligned}\delta^\star_\alpha(\phi_1 \star \phi_2) &= \mathrm{i} m_\star\{\Delta_{\mathcal{F}}(\delta^\star_\alpha)\phi_1 \otimes \phi_2\} \\ &= \mathrm{i}(\alpha\phi_1) \star \phi_2 + \mathrm{i}\phi_1 \star (\alpha\phi_2) \\ &\quad + \mathrm{i}\sum_{k=1}^{\infty} \tfrac{1}{k!}\left(\tfrac{-\mathrm{i}}{2}\right)^k \theta^{\mu_1\nu_1}\dots\theta^{\mu_k\nu_k}\Big[(\partial_{\mu_1}\dots\partial_{\mu_k}\alpha)\phi_1 \star (\partial_{\nu_1}\dots\partial_{\nu_k}\phi_2) \\ &\qquad + (\partial_{\mu_1}\dots\partial_{\mu_k}\phi_1) \star (\partial_{\nu_1}\dots\partial_{\nu_k}\alpha)\phi_2\Big]\,.\end{aligned} \tag{3.13}$$

Of course, this formalism can be readily used to include gauge fields as well. As usual, the field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu\right] \,, \tag{3.14}$$

which transforms covariantly as $\delta^\star_\alpha F_{\mu\nu} = \mathrm{i}g\,[\alpha, F_{\mu\nu}]$. For the Groenewold-Moyal case, the action reads

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu} \star F^{\mu\nu} \,. \tag{3.15}$$

There is a remarkable difference to the non-twisted approach: Starting with a Lie algebra valued connection, twisted gauge transformations close in the Lie algebra. However, the consistency of the equations of motion of (3.15) require the introduction of additional new vector potentials. The number of the new degrees of freedom is representation dependent, but remains finite for finite dimensional representations.

To summarize, the main idea of this approach is to extend symmetry transformations, (co-)products, etc. by twists $\mathcal{F}$ in a consistent way. This approach can be generalized to x-dependent star products if these products can be expressed in terms of a twist $\mathcal{F}$ as

$$(f \star g)(x) = m(\mathcal{F}^{-1} f \otimes g) \,. \tag{3.16}$$

3.3 The fuzzy sphere

The fuzzy sphere has been introduced by J. Madore (1990). Its generators satisfy linear commutation relations

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\frac{\theta}{r}\epsilon_{ijk}\hat{x}_k \,, \qquad i,j,k \in \{1,2,3\} \,, \tag{3.17}$$

where $r^2\mathbb{1} = (\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)$ and $r \in \mathbb{R}$ is the radius of the sphere. The objects $\hat{R}_i = \frac{r}{\theta}\hat{x}_i$ obviously satisfy the $SU(2)$ algebra relations. We may choose an irreducible representation with spin j, so that the generators $\hat{R}_i$ as well as $\hat{x}_i$ are $N \times N$ matrices with $N = 2j+1$. The parameter θ is fixed by the quadratic Casimir relation

$$\theta = \frac{r^2}{\sqrt{j(j+1)}} \,. \tag{3.18}$$

It is related to the elementary area on the sphere, which becomes obvious after a rescaling

$$\theta' = \frac{r^2}{j+\frac{1}{2}} = \frac{4\pi r^2}{2\pi N} \,. \tag{3.19}$$

The space algebra (3.17) is equipped with a differential calculus. Since we are dealing with matrix algebras, all derivations are inner. The differentials $\hat{\partial}_i$ satisfy the same algebra as the coordinates:

$$\left[\hat{\partial}_i, \hat{\partial}_j\right] = \frac{1}{r}\epsilon_{ijk}\hat{\partial}_k \,, \tag{3.20}$$

and therefore they can be represented as

$$\hat{\partial}_i = -\frac{\mathrm{i}}{\theta}\hat{x}_i \,. \tag{3.21}$$

The adjoint action of $\hat{R}_i$ on a function $\hat{f}$ generates rotations of $\hat{x}_i$, hence

$$\hat{L}_i \hat{f} = \left[\hat{R}_i, \hat{f}\right] , \tag{3.22}$$

where $\hat{L}_i$ denote the generators of angular momentum. The integral over the fuzzy sphere is given by the trace with respect to the matrix space,

$$\int \hat{f} = \frac{4\pi r^2}{N} \mathrm{Tr}\, \hat{f} . \tag{3.23}$$

The constant prefactor ensures the correct commutative limit, which is accomplished by keeping r fixed and taking $\theta \to 0$ (corresponding to $j \to \infty$). The non-commutative Moyal plane is recovered in the limit $r \to \infty$ and keeping θ fixed (corresponding to $j \to \frac{r^2}{\theta}$).

Gauge fields are introduced via the covariant derivatives $\hat{D}_i = \hat{\partial}_i + \mathrm{i}\hat{A}_i$, where $\hat{A}_\alpha$ are Hermitian $N \times N$ matrices. The field strength is given by

$$\mathrm{i}\hat{F}_{ij} = \left[\hat{D}_i, \hat{D}_j\right] - \frac{\epsilon_{ijk}}{r} \hat{D}_k , \tag{3.24}$$

and gauge transformations read

$$\hat{D}_i' = u\hat{D}_i u^{-1} , \qquad \hat{F}_{ij}' = u\hat{F}_{ij} u^{-1} , \tag{3.25}$$

where u is a $U(N)$ matrix. The restriction of the gauge field to the sphere is expressed as $\sum_i \hat{X}_i^2 = r^2$ leading to

$$\hat{x}_i \hat{A}_i + \hat{A}_i \hat{x}_i - \theta \hat{A}_i^2 = 0 , \tag{3.26}$$

where covariant coordinates $\hat{X}_i = \mathrm{i}\theta\hat{D}_i = \hat{x}_i - \theta\hat{A}_i$ are used. Hence, the action for the gauge field is given by

$$S = \frac{4\pi r^2}{N} \mathrm{Tr}\, \hat{F}_{ij} \hat{F}_{ij} . \tag{3.27}$$

A complex scalar field $\hat{\Phi}$ can be coupled to a gauge theory using the minimal coupling

$$S[\hat{\Phi}, \hat{A}] = \frac{4\pi r^2}{\theta^2 N} \mathrm{Tr} \left(\left[\hat{X}_i, \hat{\Phi}\right] \left[\hat{\Phi}, \hat{X}_i\right] + \theta^2 V(\hat{\Phi}) \right) . \tag{3.28}$$

3.4 κ-deformed spaces

As an alternative to (3.17) one may prefer to consider a linear commutation relation that is compatible with a deformed version of Poincaré symmetry. The most general such commutation relation is given by

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i} \left(a^\mu \delta^\nu_\sigma - a^\nu \delta^\mu_\sigma \right) \hat{x}^\sigma , \tag{3.29}$$

where a^μ is a constant 4-vector "pointing into the direction of non-commutativity". Its components also play the role of Lie algebra structure constants. In fact, a recent new motivation for studying such κ-deformed space-times comes from "double special relativity"

(DSR), where in addition to the speed of light c a second invariant parameter κ of mass dimension 1 is introduced.

Of course, in Euclidean spaces all directions are equivalent. We choose the n-direction, i.e. $a^\mu = \kappa^{-1}\delta^{n\mu}$, where κ is the parameter which gives its name to this approach. Now, let us define derivatives on this κ-Euclidean space. We introduce them by finding a deformed Leibniz rule compatible with the algebra relations (3.29), i.e.

$$\begin{aligned} \hat\partial_n \hat x^i &= \hat x^i \hat\partial_n\,, & \hat\partial_n \hat x^n &= 1 + \hat x^n \hat\partial_n\,,\\ \hat\partial_i \hat x^j &= \delta_i^j + \hat x^j \hat\partial_i\,, & \hat\partial_i \hat x^n &= \left(\hat x^n + \frac{\mathrm{i}}{\kappa}\right)\hat\partial_i\,. \end{aligned} \tag{3.30}$$

Note that these relations are not unique, though. The commutator of derivatives compatible with (3.29) is given by $[\hat\partial_\mu, \hat\partial_\nu] = 0$. The Leibniz rule for non-commutative functions reads

$$\hat\partial_i \hat f(\hat x) = (\hat\partial_i \hat f(\hat x)) + \hat f(\hat x^i, \hat x^n + \mathrm{i}/\kappa)\,\hat\partial_i\,, \tag{3.31}$$

and the derivative $\hat\partial_n$ satisfies the ordinary Leibniz rule. The co-product of the translation generators reads

$$\Delta\hat\partial_n = \hat\partial_n \otimes \mathbb{1} + \mathbb{1}\otimes\hat\partial_n\,, \qquad \Delta\hat\partial_i = \hat\partial_i\otimes\mathbb{1} + e^{\frac{\mathrm{i}}{\kappa}\hat\partial_n}\otimes\hat\partial_i\,. \tag{3.32}$$

Let us now introduce the star product using a symmetrical ordering. Considering the "non-commutative direction" $n = 1$, it is given by

$$(f\star g)(x) = \int d^4k\, d^4p\, \tilde f(k)\tilde g(p)\, e^{\mathrm{i}(\omega_k+\omega_p)x^1} e^{\mathrm{i}\vec x(\vec k e^{\frac{\omega_p}{\kappa}} A(\omega_k,\omega_p) + \vec p A(\omega_p,\omega_k))}\,, \tag{3.33}$$

where $k = (\omega_k, \vec k)$, $\vec x = (x^2, x^3, x^4)$, and

$$A(\omega_k,\omega_p) \equiv \frac{\frac{1}{\kappa}(\omega_k+\omega_p)}{e^{\frac{1}{\kappa}(\omega_k+\omega_p)}-1}\,\frac{\kappa\left(e^{\frac{\omega_k}{\kappa}}-1\right)}{\omega_k}\,. \tag{3.34}$$

In symmetrical ordering, the action of the deformed derivatives on commutative functions (denoted by $\partial^\star$) can be expressed in terms of the usual derivatives

$$\partial_i^\star f(x) = \partial_i\, e^{\frac{\mathrm{i}}{\kappa}\partial_n}\, f(x)\,, \qquad \partial_n^\star f(x) = \partial_n\, f(x)\,. \tag{3.35}$$

3.5 q-deformation

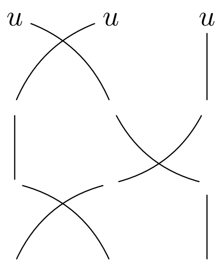
In this section, we want to discuss the construction of gauge theory on q-deformed spaces. First recall the commutator relation Eqn. (1.26):

$$\left[\hat x^i, \hat x^j\right] = \left(\frac{1}{q}\hat R^{ij}_{kl} - \delta^i_l\delta^j_k\right)\hat x^k \hat x^l\,. \tag{3.36}$$

These spaces are representations of quantum groups, Hopf algebras which in addition possess one additional structure: the so-called $\hat R$-matrix. Let $\hat u^k_m$ be the generators of the Hopf algebra. Then the $\hat R$-matrix deforms the multiplication in the algebra:

$$\hat R^{ij}_{kl}\hat u^k_m \hat u^l_n = \hat u^i_k \hat u^j_l \hat R^{kl}_{mn}\,, \tag{3.37}$$

where $\hat{R}$ itself is a solution of the Yang-Baxter equation, i.e. $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$, with ${R_{12}}^{ijk}_{lmn} := \hat{R}^{ij}_{lm}\,\delta^k_n$ and ${R_{23}}^{ijk}_{lmn} := \hat{R}^{jk}_{mn}\,\delta^i_l$. This relation is also called *braid equation* because it can be graphically represented by a braid. For example, the l.h.s. of the braid equation can be depicted as (where crossings denote two u contracted by an $\hat{R}$-matrix):



There exists a whole graphical apparatus to deal with the braid group. Especially, S. Majid pushed this mathematical approach, which was also applied to gauge theory.

Quantum spaces with generators $\hat{x}^i$ are representations of the respective quantum group. The algebra relations of the generators are consistently given by

$$P_{-}{}^{ij}_{kl}\hat{x}^k\hat{x}^l = 0\,, \tag{3.38}$$

where P_- is the q-deformed antisymmetric projector, generalizing the commutator, from the projector decomposition of the $\hat{R}$-matrix of the respective quantum group. Considering the quantum groups $GL_q(N)$ or $SL_q(N)$, we have the following decomposition

$$\hat{R} = qP_+ - q^{-1}P_-\,, \tag{3.39}$$

and in the case of $SO_q(N)$,

$$\hat{R} = qP_+ - q^{-1}P_- + q^{1-N}P_0\,, \tag{3.40}$$

with self-explaining notation. In the commutative limit $q \to 1$, we obtain $\hat{R}^{ij}_{kl} \to \delta^i_l\delta^j_k$ — cp. Eqn. (3.36). A covariant (with respect to the action of the quantum group) differential calculus also exists and can be defined by the following relations:

$$\hat{P}_{-}{}^{ij}_{kl}\hat{\partial}_i\hat{\partial}_j = 0\,, \qquad \hat{\partial}_i\hat{x}^j = \delta^j_i + q^{\pm 1}\hat{R}^{\pm 1}{}^{jl}_{ik}\hat{x}^k\hat{\partial}_l\,. \tag{3.41}$$

In 2002, S. Schraml computed the Seiberg-Witten map up to first order in $h := \ln q$, and with respect to a normal ordered star product for a $SL_q(2)$-symmetric quantum space, the so-called *Manin plane*. The same approach was also studied by others where gauge theory is formulated on Euclidean q-deformed 2-dimensional spaces generated by $\hat{z}, \hat{\bar{z}}$ with relation $\hat{z}\hat{\bar{z}} = q^2\hat{\bar{z}}\hat{z}$, which is covariant under the quantum group $E_q(2)$. In order to formulate an action, one uses the Hermitian star product

$$(f \star g)(\zeta, \bar{\zeta}) = m \circ e^{h(\zeta\partial_\zeta \otimes \bar{\zeta}\partial_{\bar{\zeta}} - \bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)} \tag{3.42}$$

and the integration measure $\mu = \frac{1}{\zeta\bar{\zeta}}$, such that

$$\int d\zeta\, d\bar{\zeta}\, \mu(f \star g)(\zeta, \bar{\zeta}) = \int d\zeta\, d\bar{\zeta}\, \mu(g \star f)(\zeta, \bar{\zeta}) = \int d\zeta\, d\bar{\zeta}\, \mu g(\zeta, \bar{\zeta}) \cdot f(\zeta, \bar{\zeta})\,. \tag{3.43}$$

This property of the integral implies that a variational calculus can be applied, and the gauge invariant action reads

$$S = \int d\zeta \, d\bar{\zeta} \, \mu \widehat{F}_{12} \star \widehat{F}_{12} \,, \tag{3.44}$$

where $\widehat{F}_{12}$ is the q-deformed non-Abelian field strength.

Due to the involved structure in the quantum group case, not many results are available, and the conducted work is mainly restricted to the formulation of models and to the discussion of rather general properties. The computation of Feynman rules and explicit perturbative (one-loop) calculations are still missing.

3.6 Yang-Mills matrix models

By considering matrix models of Yang-Mills type, a different interpretation of the origin of the UV/IR mixing in non-commutative gauge models can be given as we will now briefly discuss. We start with the Yang-Mills matrix model action

$$S_{YM} = -\mathrm{Tr}\left[X^a, X^b\right]\left[X^c, X^d\right]\eta_{ac}\eta_{bd} \,, \tag{3.45}$$

where η_{ab} denotes some D dimensional embedding space. The X^a are Hermitian matrices acting on a Hilbert space $\mathcal{H}$. In the simplest case, these matrices represent generalized "coordinates", and if some of them are functions of the others, in the semi-classical limit $X \sim x$ one can interpret these as defining the embedding of a $2n$ dimensional submanifold $\mathcal{M}^{2n} \in \mathbb{R}^D$ equipped with a non-trivial induced metric

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab} \,, \tag{3.46}$$

via pull-back of η_{ab}. Projectors on the tangential/normal bundle of $\mathcal{M}$ are defined as $\mathcal{P}_T^{ab} = g^{\mu\nu}\partial_\mu x^a \partial_\nu x^b$ and $\mathcal{P}_N^{ab} = \eta^{ab} - \mathcal{P}_T^{ab}$. The situation is illustrated in Figure 3.1.

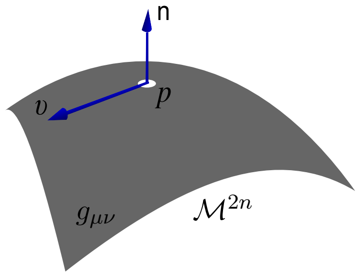


Figure 3.1: The submanifold $\mathcal{M}^{2n}$ with induced metric.

This submanifold could then e.g. be our (non-commutative) 4-dimensional space-time $\mathcal{M}^4$ endowed with a Poisson structure $\theta^{\mu\nu} \sim -\mathrm{i}\,[X^\mu, X^\nu]$. In fact, the Poisson structure $\theta^{\mu\nu}$ (assuming it is non-degenerate) and the induced metric $g_{\mu\nu}$ combine to an "effective" metric

$$G^{\mu\nu} = e^{-\sigma}\theta^{\mu\rho}\theta^{\nu\sigma}g_{\rho\sigma} \,, \qquad e^{-\sigma} \equiv \frac{\sqrt{\det \theta^{-1}_{\mu\nu}}}{\sqrt{\det G_{\rho\sigma}}} \,, \tag{3.47}$$

which is the one that is actually "felt" by matter fields. Furthermore, the matrix model action (3.45) is invariant under the gauge symmetry $X^\mu \to uX^\mu u^{-1}$, where $u \in U(\infty)$, as well as under global rotation and translation symmetries.

It is remarkable that within the matrix model framework four space-time dimensions, i.e. $\mu, \nu \in \{0,1,2,3\}$, play a very special role: From the definition of the effective metric (3.47) follows, that if $2n = 4$, one has $\det G_{\mu\nu} = \det g_{\mu\nu}$. This means that the special class of geometries where $G_{\mu\nu} = g_{\mu\nu}$ (which incidentally corresponds to a self-dual symplectic form $\theta^{-1}_{\mu\nu}$) is a solution of the model. Furthermore, in the 4-dimensional case the Poisson tensor $\theta^{\mu\nu}$ does not enter the Riemannian volume element, which turns out to stabilize flat space.

In order to make things clearer, consider a scalar field ϕ on $\mathcal{M}^4$ in the semi-classical limit where $X^a \sim x^a$ are mere coordinates: In order to preserve gauge invariance, the kinetic term must have the form

$$\begin{aligned}
S[\phi] &= -\mathrm{Tr}\,[X^a,\phi]\,[X^c,\phi]\,\eta_{ac} \sim \int d^4x \sqrt{\det \theta^{-1}_{\mu\nu}}\,\{x^a,\phi\}_{\mathrm{PB}}\{x^c,\phi\}_{\mathrm{PB}}\eta_{ac} \\
&= \int d^4x \sqrt{\det \theta^{-1}_{\mu\nu}}\,\theta^{\mu\nu}\partial_\mu x^a \partial_\nu\phi\,\theta^{\rho\sigma}\partial_\rho x^c \partial_\sigma \phi\,\eta_{ac} \\
&= \int d^4x \sqrt{\det G_{\mu\nu}}\,G^{\nu\sigma}\partial_\nu\phi\partial_\sigma\phi\,,
\end{aligned} \tag{3.48}$$

cf. Eqn. (3.28). This semi-classical effective action obviously describes a scalar field on a 4-dimensional space-time with metric $G_{\mu\nu}$, and if $G_{\mu\nu} = g_{\mu\nu}$ it becomes independent of the Poisson tensor $\theta^{\mu\nu}$ (in this approximation), as claimed above.

In a further step, it is also possible to add $U(N)$ gauge fields A to the matrix model. To show this, we start with the equations of motion of the matrix model action (3.45):

$$\left[X^a, \left[X^b, X^c\right]\right]\eta_{ab} = 0\,. \tag{3.49}$$

For every solution X^c of this equation, $X^c \otimes \mathbb{1}_N$ is a solution[1] as well. The fluctuations A_μ in the submanifold $\mathcal{M}^4$ around such a background can be parametrized by

$$X^a = \bar{X}^a + \mathcal{A}^\mu(\bar{X})\,, \qquad\qquad \mathcal{A}^\mu = -\theta^{\mu\nu}A_\nu(\bar{X})\,, \tag{3.50}$$

where the A_μ are some $U(N)$ valued fields[2]. The effective matrix model action then describes gauge fields in a gravitational background. However, though inseparable, the $U(1)$ and the $SU(N)$ subsectors play very different roles: In fact, the $U(1)$ fields contribute *only* to the gravitational sector, i.e. they represent geometrical degrees of freedom. This means, that within the matrix model framework, non-commutative $U(N)$ gauge field theory describes $SU(N)$ fields coupled to gravity.

Finally, matrix valued fermions Ψ can be added as well. For example, the IKKT model[3], whose action is given by

$$\begin{aligned}
&S_{\mathrm{IKKT}} = \mathrm{Tr}\left(\left[X^a, X^b\right][X_a, X_b] + \bar{\Psi}\not{D}\Psi\right)\,, \\
&\not{D}\Psi := \gamma_a\,[X^a,\Psi]\,, \qquad\qquad \{\gamma_a,\gamma_b\} = 2\eta_{ab}\,,
\end{aligned} \tag{3.51}$$

is supersymmetric if Majorana-Weyl spinors ($\Psi = \mathcal{C}\bar{\Psi}^T$) are considered, and it is in fact expected to be renormalizable. Hence, below the SUSY breaking scale, this model may provide a good description of quantum gravity coupled to matter.

[1] One can interpret such a solution as N coinciding branes.

[2] Notice also the similarity to the covariant coordinates we introduced in Eqn. (2.52).

[3] In fact, the IKKT model was originally proposed as a non-perturbative definition of type IIB string theory.

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